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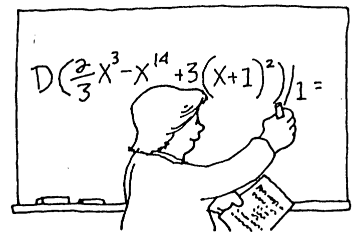

A Classroom Capsule is a short article that contains a new insight on a topic taught in the earlier years of undergraduate mathematics. Please submit manuscripts prepared according to the guidelines on the inside front cover to Tom Farmer.

Bond Duration: An Application of Calculus

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One important application of calculus in finance that has not yet appeared in calculus textbooks is that of the *duration* of a bond, which is a measure of the sensitivity of the bond's price to changes in interest rates. Most people are aware that the value or market price of a fixed-income investment such as a bond decreases as interest rates rise and increases as interest rates fall. Moreover, long-term bonds such as 30-year treasuries are more sensitive to interest rate fluctuations than short-term investments such as 2-year treasuries. These characteristics were illustrated during 1994 when the Federal Reserve Board raised short-term interest rates six times. As a result of these actions, most bond mutual funds showed negative total rates of return over this period, much to the dismay of their investors; the funds whose portfolios consisted of long-term bonds suffered the deepest declines.

The price of a fixed income investment is defined to be the present value of all future payments. Thus, when interest is compounded continuously, the price of a bond that pays n future payments (or "coupons") is given, as a function of the market interest rate i, by the equation

$$P(i) = \sum_{k=1}^{n} C_k e^{-it_k}, \tag{1}$$

where C_k represents the kth payment paid by the bond at time t_k.

The inadequacy of *maturity* (the length of time until the last payment is made) as a measure of risk is illustrated in Figure 1, which shows the price of two bonds that mature in five years, as functions of i. The upper graph represents a bond paying a coupon of \$60 per year for five years and having a face value of \$700; the lower graph represents a zero-coupon bond (no periodic payments) having a face value of \$1,000 at the end of five years. If the periodic payments were not reinvested, the two bonds would be equally attractive to an investor. However, in practice, the owner of the coupon-paying bond can reinvest each of the payments to produce additional income and thereby make it more valuable than the zero-coupon bond. In addition, the graphs illustrate clearly that the price of the zero-coupon bond is more sensitive to interest rate fluctuations—its price drops more for a given increase in i. This means that an investor who may have to sell a bond before maturity should consider the higher risk associated with the possible

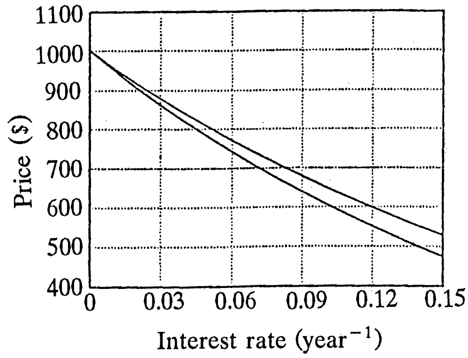

Figure 1

drop in price of this type of bond if interest rates rise. The duration of a bond helps to quantify this risk. It also helps to quantify the opportunity for financial gain when interest rates decline.

The duration, D, of a bond is defined as (the negative of) the relative rate of change of price with respect to interest rate; that is,

$$D = -\frac{P'(i)}{P(i)}. \tag{2}$$

Since $P'(i)$ is negative, the negative sign in equation (2) ensures that D will have positive values. If interest is given as an annual rate, so its units are years^{-1}, then D is measured in years. Equation (2) says that if we have a bond whose duration is 10 years, then an interest rate increase of half a point ($\Delta i = \frac{1}{2}\%$) must be accompanied by a relative change in bond price $\Delta P/P \approx -(10)(\frac{1}{2}\%) = -5\%$, i.e., a decrease in price of about 5%.

Figure 2 shows the graph of a hypothetical bond's price as a function of the interest rate, together with the graph of the tangent line at $i = 0.10$. Using only the information contained in the graph, students are asked to determine the duration of this bond when $i = 0.10$. The graph shows that $P(10) = \$75$ and the slope of the tangent line shows that $P'(10) = -1500$ \$/year, yielding $D = 20$ years. I have found this kind of example useful because it focuses on the essence of the concept without forcing the student to carry out any analysis. In addition, it reinforces

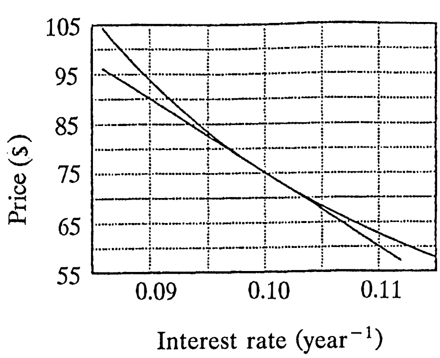

Figure 2

graph reading skills; most students are quite surprised that the absolute value of $P'(10)$ is so large.

Note that if $n = 1$ in equation (1), so the bond is a zero-coupon bond that pays a fixed amount C after T years, then (2) becomes $D = T$ years; that is, duration is the same as maturity in this case. On the other hand, by combining (1) and (2) in the general case, duration can be expressed as a weighted sum of the times $t_1, \dots, t_n$. In this form,

$$D = \sum_{k=1}^{n} \left(\frac{C_k e^{-it_k}}{P(i)} \right) t_k \tag{3}$$

represents the weighted average maturity of n separate zero-coupon bonds. This explains how the coupons serve to reduce the duration of a bond and, hence, to make its price less sensitive to changes in interest rates.

The concept of duration as given in (3) was first developed by Macauley [**5**] for the discrete compounding situation; it was later used by Hopewell and Kaufman [**4**] to explain the relationship between volatility in bond prices and maturity. Widespread use of the concept within the financial community did not occur until the 1980s. The increased importance of duration as a measure of a bond's risk is indicated by the appearance of the term in the business press [**1**], [**3**] and monthly newsletters which mutual fund advisory services send to their clients [**2**]. By incorporating this concept in appropriate courses we can give our students some useful knowledge and also convince them that mathematics is playing an increasingly important role in a wide range of disciplines.

References

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The Falling Ladder Paradox

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Anyone who has studied calculus has probably solved the classic *falling ladder* problem of related rates fame:

> A ladder L feet long leans against a vertical wall. If the base of the ladder is moved outwards at the constant rate of k feet per second, how fast is the tip of the ladder moving downwards?

The standard solution model for this problem is to assume that the tip of the ladder slips downward, maintaining contact with the wall until impact at ground level, so that if the base and tip of the ladder at any time t have coordinates