2. The normalizer, $N_G(H)$, is the union of all the right cosets Ha_i such that the table reveals $a_i^{-1}Ha_i = H$ (the entries in the ith diagonal block are exactly the elements of H).
3. Let m be the number of diagonal blocks consisting of the elements of H, then mk is the order of $N_G(H)$, m is the index of H in $N_G(H)$, and $s - m$ is the index of $N_G(H)$ in G.

Getting Normal Probability Approximations without Using Normal Tables

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Introduction

Using the normal distribution to approximate probabilities involving sums of discrete random variables is standard fare for beginning calculus-based statistics courses. These approximations are typically made with the aid of a normal probability table. In this paper, we discuss an alternative approach using the probability density function (pdf) for the normal distribution.

Exposing students to the alternative approach is well worth the minimal class time it requires. The technique is not only easier to apply in certain approximation situations than the standard method, but also gives students a different slant on random variables, probabilities, and areas.

The Normal PDF Approximation

Suppose that we have a sum of n independent discrete random variables, denoted by S, and that S takes consecutive integer values. Let $\mu = \mathrm{E}(S)$ and $\sigma^2 = \mathrm{Var}(S)$. Appealing to the Central Limit Theorem we can argue that for a sufficiently large value of n, S has an approximately normal distribution. Using the standard continuity correction, we get

$$\mathrm{P}(S = s) \approx \mathrm{P}\big(s - 0.5 < \mathrm{normal}\big(\mu, \sigma^2\big) < s + 0.5\big).$$

Viewing this approximating probability as an integral, we calculate a midpoint approximation using a single subinterval and get

$$\mathrm{P}(S = s) \approx \frac{1}{\sqrt{2\pi}\,\sigma} e^{-(s-\mu)^2/2\sigma^2}.$$

We call this the *normal pdf approximation.*

It should be noted that this approximation is far from new. Similar approximations preceded not only the Central Limit Theorem, but also the normal distribution itself. De Moivre derived an approximation of this type for binomial probabilities directly, using a limit argument. His derivation is credited by some to be the origin of the normal distribution (see Stigler [**2**]).

The normal pdf and the normal table both give essentially the same approximation. The benefits of using the normal pdf instead of the table are that it is faster and only a calculator is needed (no table). It can often even claim an accuracy advantage in calculating normal probabilities, especially if interpolation is not used with a table calculation. Compare, for example,

$$\mathrm{P}(4.5 < \text{normal}(7,37) < 5.5) = \frac{1}{\sqrt{74\pi}} \approx 0.0621 e^{-4/74}$$

with

$$\begin{aligned}\mathrm{P}(4.5 < \text{normal}(7,37) < 5.5) &= \mathrm{P}\left(\frac{-2.5}{\sqrt{37}} < \text{normal}(0,1) < \frac{-1.5}{\sqrt{37}}\right)\\ &= \mathrm{P}(-0.4110 < \text{normal}(0,1) < -0.2466)\\ &\approx 0.4013 - 0.3409 \text{ (without interpolation)}\\ &= 0.0604.\end{aligned}$$

The true value is 0.0621.

The accuracy of the normal pdf is not a surprise here. In general, if K_2 is an upper bound on $|f^{(2)}(x)|$, then $K_2/24$ is an upper bound on $|\int_{a-0.5}^{a+0.5} f(x)\,dx - f(a)|$. (See the discussion on the midpoint approximation to integrals in Ostebee and Zorn [**1**].) For our case, it is easy to calculate that $\frac{1}{\sqrt{2\pi}\sigma^3}$ is an upper bound on the second derivative of the normal (μ, σ^2) pdf, thus getting

$$\left|\mathrm{P}\big(a - 0.5 < \text{normal}(\mu, \sigma^2) < a + 0.5\big) - \frac{1}{\sqrt{2\pi}\sigma} e^{-(a-u)^2/2\sigma^2}\right| \le \frac{1}{24\sqrt{2\pi}\sigma^3}.$$

We get, for example, that if $\sigma > 5.5$, the error bound is less than 0.0001, and if $\sigma > 2.6$, the error bound is less than 0.001.

One way to make this technique easily accessible to students when teaching the material is to point out that the midpoint approximation is being used on an integral which is one unit wide. Thus for our approximation, all we need to do is evaluate the function at the middle of the interval of interest. It could also be pointed out that unless the center of the normal curve is included in the interval of interest, the pdf will either be strictly increasing or strictly decreasing—which is a good intuitive reason one can give to the student (with an appropriate picture included in the classroom discussion) in explaining why the approximation is so accurate.

If probabilities are desired for several values of s, some streamlining can be used. For example, with $\mu = 7$, $\sigma^2 = 37$,

$$\mathrm{P}(S = 7 + d) \approx \frac{1}{\sqrt{74\pi}} e^{-d^2/74},$$

so

$$\begin{aligned}\mathrm{P}(S=8) &\approx \mathrm{P}(S=7) e^{-1/74},\\ \mathrm{P}(S=9) &\approx \mathrm{P}(S=8) e^{-3/74},\\ \mathrm{P}(S=10) &\approx \mathrm{P}(S=9) e^{-5/74},\end{aligned}$$

and so on. This makes approximate probability tables easy to construct.

Examples

Next, we give three examples showing some of the ways the normal pdf approximation could be used in a class.

Example 1. A fair coin is flipped $2n$ times. What is the smallest value of n for which

a) P(we get the same number of heads as tails) < 0.05?
b) P(we get the same number of heads as tails) < 0.01?

Solution. Usng the notation we had earlier, we take S to be the number of heads obtained in $2n$ flips. We get $\mu = n$, $\sigma^2 = n/2$, and

$$\text{P(we get the same number of heads as tails)} = P(S = n) \approx 1/\sqrt{n\pi}.$$

a) We set $1/\sqrt{n\pi} < 0.05$ and get $n > 127.3$. The smallest integer n satisfying this is $n = 128$. (256 flips!). (An exact calculation gives that with $n = 127$, $P(S = 127) = 0.05001$, with $n = 128$, $P(S = 128) = 0.04982$).
b) We proceed as in (a) and get $n = 3{,}183$. (6,366 flips!). (An exact calculation gives that with $n = 3{,}182$, $P(S = 3{,}182) = 0.01000133$, with $n = 3{,}183$, $P(S = 3{,}183) = 0.00999976$).

Example 2. Suppose that $X_1, X_2, \dots, X_{20}$ are independent, $X_i \sim$ Bernoulli $\left(p_i = \frac{29 + 2i}{100}\right)$. Let $S = \sum_{i=1}^{n} X_i$. (Here S denotes the total number of successes obtained in a series of independent trials where the probability of success varies from trial to trial.) Give an approximate probability table for S, for $s = 2, \dots, 10$.

Solution. Here, we calculate $\mu = 10$, $\sigma^2 = 4.734$, and get $P(S = s) \approx \frac{1}{\sqrt{2\pi}\sqrt{4.734}} e^{-(s-10)^2/9.468}$. In the table below, we have included exact probabilities for comparison.

s	True $P(S = s)$	Normal PDF Approximation
2	0.0001	0.0002
3	0.0008	0.0010
4	0.0038	0.0041
5	0.0131	0.0131
6	0.0346	0.0338
7	0.0721	0.0709
8	0.1207	0.1202
9	0.1639	0.1650
10	0.1813	0.1834

Example 3. There are 20 girls and 30 boys in Group 1, 25 girls and 25 boys in Group 2, and 10 girls and 10 boys in Group 3. If 10 children are randomly chosen from each group and S denotes the total number of girls chosen, give an approximation probability table for S, for $s = 7, \dots, 21$.

Solution. Here, S is the sum of three hypergeometric random variables. Using the mean and variance formulas for the hypergeometric distribution, we get the mean of S is $\mu = 14$ and the variance of S is $\sigma^2 = 5.316$. $P(S = s) \approx \frac{1}{\sqrt{2\pi}\sqrt{5.316}} e^{-(s-14)^2/10.632}$
In the table that follows, we have included exact probabilities for comparison.

s	True $P(S=s)$	Normal PDF Approximation
7	0.0015	0.0017
8	0.0056	0.0059
9	0.0164	0.0165
10	0.0389	0.0384
11	0.0752	0.0742
12	0.1197	0.1188
13	0.1576	0.1575
14	0.1721	0.1730
15	0.1563	0.1575
16	0.1182	0.1188
17	0.0744	0.0742
18	0.0388	0.0384
19	0.0168	0.0165
20	0.0060	0.0059
21	0.0017	0.0017

Classroom Experience. In the fall of 1997, the normal table and the normal pdf approximations were demonstrated to students in a calculus-based probability class. The final exam for the class included the following problem: Let $S = X_1 + \cdots + X_{150}$, where the X's are independent, $P(X_i = -1) = P(X_i = 0) = P(X_i = 1) = 1/3$. Find a normal approximation to $P(S = 5)$. All eight students gave correct solutions. Two used a normal table approximation. Five chose a normal pdf approximation. The remaining student used a pdf approximation, then checked his answer with a normal table calculation.

Overall, the introduction of the normal pdf approximation method seemed appropriate for the class and enriching for the students. Its introduction will be included the next time the class is taught.

References

1. Ostebee, A. and P. Zorn, *Calculus from Graphical, Numerical, and Symbolic Points of View.* Harcourt Brace, 1997.
2. Stigler, S., *The History of Statistics.* Harvard University Press, 1987.

Normal Lines and Curvature

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Let C be a differentiable curve whose equation is $y = f(x)$. It is standard practice in calculus textbooks to introduce the tangent line to C at the point $P(a, f(a))$ by considering a nearby point $Q(a+h, f(a+h))$, where $h \neq 0$, then considering the slope m_{PQ} of the secant line PQ and defining the tangent line to C at P to be the line containing $P(a, f(a))$ and having slope $m = \lim_{h \to 0} m_{PQ}$.