Thus if the data set contains n elements, Chebyshev's theorem guarantees that at most n/k^2 of these data values lie farther than k standard deviations away from μ. Suppose for a moment that we have a data set of n elements for which this is not the case, for some particular k. Call the elements that lie farther than k standard deviations away from the mean *outsiders*. Since every outsider lies at a distance greater than $k\sigma$ from the mean μ, each tile associated with an outsider has area greater than $(k\sigma)^2 = k^2\sigma^2$. We have assumed the existence of more than n/k^2 outsiders, so it follows that the combined area of the outsider tiles must exceed $(n/k^2)(k^2\sigma^2) = n\sigma^2$. But this is impossible—the total area of *all* the tiles is $n\sigma^2$. This contradiction proves Chebyshev's theorem.

In short, Chebyshev's theorem says all data sets are xenophobic—they cannot allow too many outsiders, lest the outsiders occupy too much of the total tiled area.

Pizza Combinatorics

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Customer: So what's this new deal?
Pizza Chef: Two pizzas.
Customer: [*Towards four-year-old boy*] Two pizzas. Write that down.
Pizza Chef: And on the two pizzas choose any toppings—up to five [*from the list of* 11 *toppings*].
Older Boy: Do you...
Pizza Chef: ...have to pick the same toppings on each pizza? No!
Four-year-old Math Whiz: Then the possibilities are endless.
Customer: What do you mean? Five plus five are ten.
Math Whiz: Actually, there are 1,048,576 possibilities.
Customer: Ten was just a ballpark figure.
Old Man: You got that right.

On November 8, 1993, this popular commercial, "Math Whiz," first aired on national television. Probably some viewers dimly recalled from their mathematical studies that the large number of possibilities has something to do with permutations, factorials, combinations, or some other long-forgotten technique, but perhaps only the authors of this article were so intrigued that they investigated whether the four-year-old "math whiz" was actually correct in his calculations.

The boy presumably reached his conclusion using the calculation

$$\left[\binom{11}{5}+\binom{11}{4}+\binom{11}{3}+\binom{11}{2}+\binom{11}{1}+\binom{11}{0}\right]^2=[1024]^2=1{,}048{,}576,$$

where each term in the sum represents the number of ways to choose a given number of toppings, up to five, from the 11 available toppings. The result was then squared, since the toppings on the two pizzas need not be the same. However, this analysis is flawed. There are indeed 1024 different pizzas possible, but only $\binom{1024}{2}+1024$ ways to order a pair of them, counting orders for two *different* pizzas and orders for two *identical* pizzas separately. This results in 524,800 possibilities for two pizzas.

Further consideration of the pizza ordering process leads to other interesting counting problems. What if toppings can be repeated in ordering a pizza—allowing orders such as "pepperoni with double mushroom"? Suppose the 11 toppings are pepperoni (P), mushroom (M), green pepper (GP), onions (O), ham (H), Canadian bacon (CB), bacon (B), ground beef (GB), sausage (S), black olives (BO), and anchovies (A). The order form simply lists symbols for the possible toppings, separated by vertical bars:

$$P|M|GP|O|H|CB|B|GB|S|BO|A$$

and the pizza chef just marks an X over the symbol for each topping you order. If you want double pepperoni, counting as two toppings, then two X's are placed over the P symbol slot.

Allowing multiple amounts of a topping, how many different pizzas are possible? The easiest way of tackling this problem is to consider the process of filling out the order form. For double pepperoni, onions, and double sausage on a pizza, the menu pad would look like this:

$$XX|\ |\ |X|\ |\ |\ |\ |XX|\ |$$

(omitting the topping symbols and the blank spaces). Similarly,

$$XXX|\ |\ |\ |\ |\ |\ |X|\ |\ |X$$

would represent triple pepperoni, ground beef, and anchovies. Each five-topping pizza, allowing multiple amounts of a topping, is thus represented by a sequence of ten bars and five X's. To specify such a pizza we just need to say which five of the 15 terms in the sequence are X's. This results in

$$\binom{15}{5}=3003$$

different pizzas with exactly five toppings. Likewise, the number of pizzas with exactly four toppings (ten bars and four X's) is

$$\binom{14}{4}=1001.$$

Continuing in this manner, the number of possibilities for one pizza, allowing multiple amounts of toppings, is

$$\binom{15}{5}+\binom{14}{4}+\binom{13}{3}+\binom{12}{2}+\binom{11}{1}+\binom{10}{0}=4368.$$

Thus, reasoning as before, the number of possible orders for two pizzas is $\binom{4368}{2}+4368=9{,}541{,}896$.

The "math whiz" should now be informed that he can order a different set of two pizzas each day for over eight million more days than he originally calculated. But wait! What if we allow "extra cheese" to count as a topping, or allow no more than doubles of any topping...?

Enjoy!

Visualizing the Group Homomorphism Theorem

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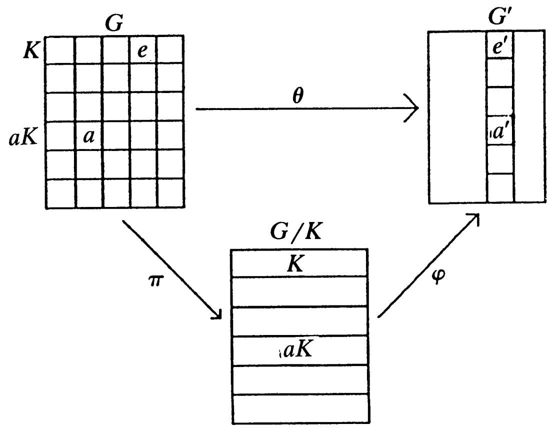
Inability to visualize the abstract concepts of group theory is one reason many students find the subject difficult. The accompanying figure should help such students visualize the cosets of a subgroup K of a finite group G, the quotient group G/K, and the fundamental group homomorphism theorem.



The group homomorphism theorem.

The groups are pictured as rectangles, rather than the usual ovals, and small squares inside G represent the group elements. The figure illustrates the partition of G into cosets of a normal subgroup K, indicating that all the cosets have the same number of members, and the order of K divides that of G. The strips forming G/K illustrate that the individual members of the quotient group are sets (the cosets of K). The diagram shows that a homomorphism $\theta\colon G \to G'$ with kernel K maps all members of a given coset of K to a single element in G', while the projection $\pi\colon G \to G/K$ maps all members of that coset to a single member of the quotient group; hence there is a one-to-one correspondence φ (a group isomorphism) between G/K and the image $\theta(G)$ (a subgroup of G'), making the diagram commute.