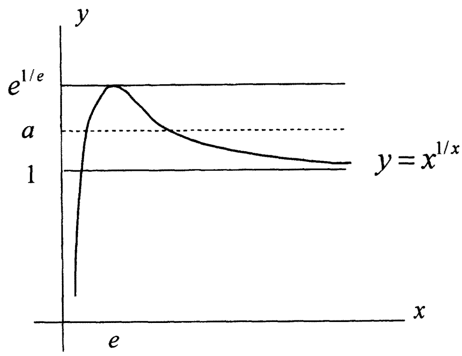
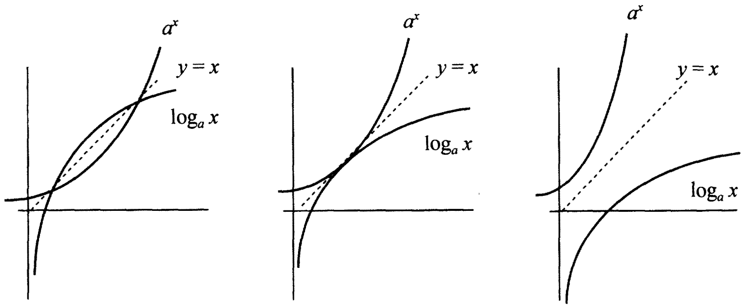


Figure 1.



$1 < a < e^{1/e}$ **Figure 2a.** $a = e^{1/e}$ **Figure 2b.** $a > e^{1/e}$ **Figure 2c.**

and exponential functions were discussed in texts, or correct graphs were given with pointers to one or more exercises for justification.

A Numerical Introduction to Partial Fractions

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This article introduces a numerical analogue to partial fraction decomposition. The reasons explaining why the numerical process works are different from those behind the algebraic procedure. (See page 150 of **[1]** for a sketch of a proof of the algebraic theorem.) However, the analogue is so similar—and so much more intuitive—that most students come away from it better equipped to face partial fraction decomposition with confidence and authority.

The decomposition of a rational expression $N(x)/D(x)$ whose numerator has degree smaller than its denominator involves three primary steps **[2]**:

(a) Express $D(x)$ as a product of its "prime" factors, say

$$D(x) = (g_1(x))^{a_1} \cdots (g_n(x))^{a_n}$$

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(b) For each $i = 1, \ldots, n$ and each $j = 1, \ldots, a_i$, let $f_{ij}(x)$ denote a polynomial of degree less than that of $g_i(x)$.

(c) Solve

$$\frac{N(x)}{D(x)} = \frac{f_{11}(x)}{g_1(x)} + \frac{f_{12}(x)}{(g_1(x))^2} + \frac{f_{1a_1}(x)}{(g_1(x))^{a_1}} + \cdots + \frac{f_{na_n}(x)}{(g_n(x))^{a_n}}$$

for each $f_{ij}(x)$.

Except in the simplest of cases, the first step identified above requires at least polynomial division, and may require a number of advanced techniques before $D(x)$ can successfully be written as a product of linear and quadratic factors. Carrying out step (c) generally involves solving a system of linear equations. Although students generally come armed with these skills, many become lost in the details of the calculations and lose sight of the goal toward which they are working. For these reasons, I offer the following introduction to the technique of partial fraction decomposition.

Rather than presenting my class with a rational expression in one variable, I begin by asking them to apply the following steps to a proper numerical fraction N/D:

(a′) Express D as a product of prime factors, say

$$D = p_1^{a_1} p_2^{a_2} \cdots p_n^{a_n}.$$

(b′) For each $i = 1, \ldots, n$ and each $j = 1, \ldots, a_i$, let k_{ij} denote an integer for which $|k_{ij}| < p_i$.

(c′) Solve

$$\frac{N}{D} = \frac{k_{11}}{p_1} + \frac{k_{12}}{p_1^2} + \frac{k_{1a_1}}{p_1^{a_1}} + \cdots + \frac{k_{na_n}}{p_n^{a_n}}$$

for each k_{ij}.

Students learned to perform step (a′) in the seventh grade, and step (c′) can usually be accomplished with a bit of trial and error. (I suggest that step (c′) be approached by first multiplying both sides of the equation by D, and then searching for solutions.) Because the skill levels required to perform these steps are minimal, students approach this exercise as a game rather than a task. Moreover, because steps (a′)–(c′) are so similar to steps (a)–(c), students learn the technique of partial fraction decomposition through the game without losing sight of the goal of the process.

Example. Let's apply steps (a′)–(c′) to $71/90$. Since $90 = 2 \times 3^2 \times 5$, we begin by writing

$$\frac{71}{90} = \frac{A}{2} + \frac{B}{3} + \frac{C}{3^2} + \frac{D}{5},$$

where A is between -1 and 1, both B and C are between -2 and 2, and D is between -4 and 4. Solving for A, B, C, and D is easier if we first multiply both sides of our equation by 90:

$$71 = 45A + 30B + 10C + 18D.$$

Students quickly recognize that $45(1) + 18(2) = 81$, and then realize that $A = 1$, $B = 0$, $C = -1$, and $D = 2$ is a solution.

We call $1/2 + 0/3 + -1/3^2 + 2/5$ a *J-decomposition* of $71/90$. In general, we call

$$\frac{k_{11}}{p_1} + \frac{k_{12}}{p_1^2} + \frac{k_{1a_1}}{p_1^{a_1}} + \cdots + \frac{k_{na_n}}{p_n^{a_n}}$$

J-decomposition of N/D whenever each k_{ij} and p_i satisfy (a′)–(c′). Some proper fractions have a unique J-decomposition—as, for example, the J-decomposition of any proper fraction with a prime denominator. However, some fractions have more than one J-decomposition. For instance, $71/90$ also has the J-decomposition

$$\frac{71}{90} = \frac{1}{2} + \frac{2}{3} + \frac{2}{3^2} + \frac{-3}{5}.$$

Characterizing the set of proper fractions that have unique J-decompositions, or developing a modification to (a′)–(c′) that would result in a unique J-decomposition for each proper fraction, would make interesting investigations.

We complete this note by proving that every proper fraction N/D has at least one J-decomposition. For positive proper fractions of the form N/p^n, observe that $k_1/p + k_2/p^2 + \cdots + k_n/p^n$ is a J-decomposition of N/p^n if and only if $N = k_1p^{n-1} + k_2p^{n-2} + \cdots + k_n$. But this simply says that $k_1k_2\cdots k_n$ is the representation of N in base p. Since every N has such a representation, it follows that

$$\frac{N}{p^n} \text{ has a } J\text{-decomposition.} \tag{1}$$

We also note the following.

$$\text{If } \frac{N}{D} \text{ has a } J\text{-decomposition, then so does } \frac{-N}{D}. \tag{2}$$

Now, assume for the purpose of induction that for fixed n, every positive proper fraction of the form $k/p_1^{a_1}\cdots p_{n-1}^{a_{n-1}}$ has a J-decomposition. We want to show that any positive proper fraction of the form $N/p_1^{a_1}\cdots p_{n-1}^{a_{n-1}}p_n^{a_n}$ also has a J-decomposition. For simplicity of notation, let $\pi_1 = p_1^{a_1}\cdots p_{n-1}^{a_{n-1}}$ and $\pi_2 = p_n^{a_n}$. Since π_1 and π_2 are relatively prime, there exist integers A and B such that $1 = A\pi_1 + B\pi_2$. It follows that

$$\frac{N}{\pi_1\pi_2} = \frac{NA}{\pi_2} + \frac{NB}{\pi_1}.$$

Using the division algorithm, we can find integers q_i, r_i for $i = 1, 2$ with $0 \le r_i < \pi_i$ such that $NA = q_2\pi_2 + r_2$ and $NB = q_1\pi_1 + r_1$. Thus,

$$\frac{N}{\pi_1\pi_2} = q_1 + q_2 + \frac{r_1}{\pi_1} + \frac{r_2}{\pi_2}. \tag{3}$$

Since q_1 and q_2 are integers, and the fractions in (3) are positive and proper, it follows that $q_1 + q_2$ is either -1 or 0. If $q_1 + q_2 = 0$, then $N/\pi_1\pi_2 = r_1/\pi_1 + r_2/\pi_2$. Since r_1/π_1 has a J-decomposition (by the induction hypothesis), we see that $N/\pi_1\pi_2 = r_1/\pi_1 + r_2/\pi_2$ does also. If $q_1 + q_2 = -1$, then $N/\pi_1\pi_2 = (r_1 - \pi_1)/\pi_1 + r_2/\pi_2$. Since $(\pi_1 - r_1)/\pi_1$ has a J-decomposition, so does $r_1 - \pi_1/\pi_1$ by (2). Therefore, $N/\pi_1\pi_2 = (r_1 - \pi_1)/\pi_1 + r_2/\pi_2$ has a J-decomposition. This proves that every positive proper fraction has a J-decomposition, and the full theorem now follows from (2).

Students in my classes respond with interest and enthusiasm to the game of J-decompositions. While playing, they develop an appreciation for, and a hint of understanding behind, the process of partial fraction decomposition. Since the proof outlined above involves an interesting application of the division algorithm, it could provide an entertaining and instructive supplement to an abstract algebra or number theory course.

References

1. Nathan Jacobson, *Basic Algebra I*, 2nd ed., W. H. Freeman, 1985.
2. Roland E. Larson, et. al., *Calculus with Analytic Geometry*, 6th ed., Houghton Mifflin, 1998.

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Euler's Theorem for Generalized Quadrilaterals

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In [1], J. B. Dence and T. P. Dence gave a proof of a theorem of Euler on convex quadrilaterals $ABCD$ (see Figure 1).

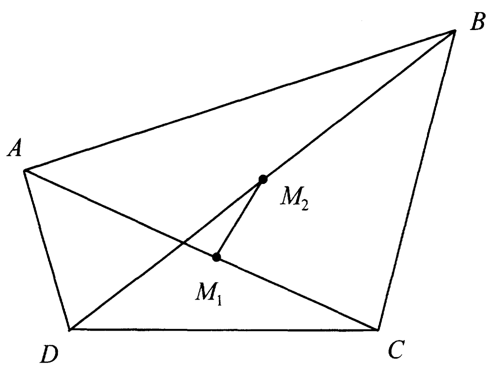

Figure 1.

Theorem. Let M_1 and M_2 denote the midpoints of AC and BD, respectively. Then

$$\overline{AB}^2 + \overline{BC}^2 + \overline{CD}^2 + \overline{DA}^2 = \overline{AC}^2 + \overline{BD}^2 + 4\overline{M_1M_2}^2.$$

(In other words. the sum of the squares of the sides is equal to the sum of the squares of the diagonals, increased by four times the square of the segment joining the midpoints of the diagonals.)

Actually, Euler's theorem is valid for a much broader class of quadrilaterals, which I refer to as generalized quadrilaterals. A *generalized quadrilateral* $ABCD$ in R^n is the figure that has A, B, C, and D (any points in R^n) as *vertices*, AB, BC, CD, DA, as *sides*, and AC and BD as *diagonals*. The vertices A, B, C, D need not be coplanar