Trigonometric Identities through Calculus

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Too much time is spent on trigonometric identities before calculus and too little time after. We routinely apply methods of calculus to re-solve precalculus problems concerned with curve sketching, areas, and volumes. But while trigonometric formulas are used to find derivatives and integrals of trigonometric expressions, we seldom show students that derivatives and integrals may also be used to verify trigonometric formulas.

For differentiable functions f, g, the identity $f(x)=g(x)$ is equivalent to the (sometimes more easily established) identity $f'(x)=g'(x)$ with $f(a)=g(a)$ for some value a. A differentiation of $f(x)=\sin^2 x+\cos^2 x$ yields $f'(x)=2\sin x\cos x - 2\sin x\cos x=0$, from which follows the identity that no one can forget. A student who differentiates both sides of $\sin 2x = 2\sin x\cos x$ verifies that $\cos 2x = \cos^2 x-\sin^2 x$.

The reader may rightly be concerned with potential circular reasoning since the derivatives of trigonometric functions depend on knowing various trigonometric identities. However this note is not an attempt to derive trigonometric formulas from first principles, but rather is a reminder that the connection between mathematical formulas *is* often circular and we can use new formulas to construct older ones. For instance, the derivative of $\sin x$ is usually found from the identity

$$\sin x-\sin y = 2\sin\left(\frac{x-y}{2}\right)\cos\left(\frac{x+y}{2}\right), \tag{1}$$

which is equivalent to

$$2\sin x\cos y = \sin(x+y)+\sin(x-y). \tag{2}$$

Taking partials with respect to x and y in (2) gives the identities $2\cos x\cos y = \cos(x+y)+\cos(x-y)$ and $-2\sin x\sin y=\cos(x+y)-\cos(x-y)$; taking the partial with respect to x (or y) in (1) leads to the formula for the cosine of a sum, whose partials produce the sine of a sum, etc.

A student recently asked me where she had made her mistake in evaluating $\int \tan x\sec^2 x\,dx$. Her answer of $\sec^2 x/2+C$ from $u=\sec x$ did not agree with the book's answer of $\tan^2 x/2+C$ from $u=\tan x$. She was pleased to learn not only that her answer was correct but that she had discovered a new "proof" of the identity $\tan^2 x+1=\sec^2 x$.

One final illustration: While working on a paper, I had the vague feeling that a needed expression $\sin^{-1}(2x/(1+x^2))$ could be put into a nicer form. After several false starts I differentiated to find that

$$\frac{d}{dx}\left(\sin^{-1}\left(\frac{2x}{1+x^2}\right)\right)=\frac{2}{1+x^2},$$

from which followed the identity $\sin^{-1}(2x/(1+x^2))=2\tan^{-1}x$. We leave for the reader the task of coming up with a rationale for making the substitution $u=2x/(1+x^2)$ in order to show that

$$\int\frac{dx}{1+x^2}=\frac{1}{2}\int\frac{du}{\sqrt{1-u^2}}=\frac{1}{2}\sin^{-1}\left(\frac{2x}{1+x^2}\right)+C.$$

(For related material see the paper by Rosenthal [Lattices of trigonometric identities, *College Mathematics Journal* 20 (1989) 232–234.])