A Nod to Bertrand Russell

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Indignor quandoque bonus dormitat Homerus.
(I ache whene'er good Homer nods.)

Horace, *Ars Poetica*, 359

Those who read the book *Religion and Science* by the genius Bertrand Russell will find the following passage about coin tossing in the chapter "Determinism" [**2**]:

> It is said (though I have never seen any good experimental evidence) that if you toss a penny a great many times, it will come heads about as often as tails. It is further said that this is not certain, but only extremely probable. You might toss a penny ten times running, and it might come heads each time. There would be nothing surprising if this happened once in $1,024$ repetitions of ten tosses, but when you come to larger numbers the rarity of a continual run of heads grows much greater. If you tossed a penny $1,000,000,000,000,000,000,000,000,000,000$ times you would be lucky if you got one series of 100 heads running. Such at least is the theory, but life is too short to test it empirically.

Russell's first sentence is misleading, for the Law of Large Numbers assures us that although the proportion of heads tends to $1/2$ with probability 1 as the number of tosses increases without bound, the absolute value of the difference between the number of heads and the number of tails tends to infinity, also with probability 1. For example, suppose you want the probability to be at least 99% that there will be at least $1,000,000$ more heads than tails (or vice versa). A simple calculation involving the normal approximation to the binomial probabilities shows that you need only toss a fair coin 6.4×10^{15} times; the probability that the proportion of heads is in the interval $(0.49999, 0.50001)$ would in this case be much greater than 99%. There is no contradiction, because 10^6 is infinitesimal with respect to 10^{15}.

Russell's next claim, that it would not be marvelous if, in $1,024$ repetitions of ten tosses, you got all heads once, is correct; in fact, the expected number of times you would get all heads is 1. Would you be "lucky," though, as Russell goes on to say, if you got one series of 100 heads running in 10^{30} tosses of a fair penny? It can be shown, without tremendous difficulty [**1**], that the average number of tosses until you get 100 heads in a row for the first time is $2(2^{100} - 1)$. Since

$$2(2^{100} - 1) \approx 2.5 \times 10^{30},$$

It would not be such extraordinary luck to get a sequence of 100 heads running in 10^{30} tosses; in fact, the probability that such a thing will happen is about 40%.

Acknowledgment. An apology is in order for the impertinence of critiquing the writings of a man whose demolition of humbug and rigmarole raised him to the highest station among modern philosophers.

References

1. Michael Barry and Anthony Lo Bello, The moment generating function of the geometric distribution of order k, *Fibonacci Quarterly* 31:2 (1993) 178–180.
2. Bertrand Russell, *Religion and Science*, Oxford University Press, 1961, p. 158.