Another Approach to a Class of Slowly Diverging Series

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The standard method for proving that each of the series

$$\sum \frac{1}{k}, \qquad \sum \frac{1}{k \ln k}, \qquad \sum \frac{1}{k \ln k \ln\ln k}, \ldots \tag{1}$$

diverges involves the integral test. In this note, we use the Mean Value Theorem to prove that $\sum_3^\infty 1/(k \ln k \ln\ln k)$ diverges. It will be clear that essentially the same argument can be used to show that each of the series in (1) diverges.

If $k \geq 3$, the Mean Value Theorem yields

$$\frac{\ln\ln\ln(k+1) - \ln\ln\ln k}{(k+1) - k} = \frac{1}{c \ln c \ln\ln c} \quad \text{for some } c \in (k, k+1).$$

Hence,

$$\frac{1}{(k+1)\ln(k+1)\ln\ln(k+1)} < \ln\ln\ln(k+1) - \ln\ln\ln k < \frac{1}{k \ln k \ln\ln k}. \tag{2}$$

Letting $k = n, n-1, n-2, \ldots, m$ $(3 \leq m < n)$ in (2) and adding, we obtain

$$\sum_{m+1}^{n+1} \frac{1}{k \ln k \ln\ln k} < \ln\ln\ln(n+1) - \ln\ln\ln m < \sum_{m}^{n} \frac{1}{k \ln k \ln\ln k}. \tag{3}$$

It now follows, taking $m = 3$ and letting $n \to \infty$ in the second inequality of (3), that $\sum_3^\infty 1/(k \ln k \ln\ln k)$ diverges.

The series in (1) are examples of slowly diverging series, each one after the first diverging more slowly than its predecessor (i.e., the ratios of their partial sums tends to zero). [See, for example, K. Knopp, *Theory and Application of Infinite Series*, Blackie & Sons, London, 1963, pp. 278–281.] An indication of how slowly $\sum_3^\infty 1/(k \ln k \ln\ln k)$ diverges may be obtained by estimating the sum of the nine million terms from the 10^6 term to the 10^7 term.

Using the first inequality of (3), with $m = 10^6 - 1$ and $n = 10^7 - 1$, we get

$$\sum_{k=10^6}^{10^7} \frac{1}{k \ln k \ln\ln k} < \ln\ln\ln 10^7 - \ln\ln\ln(10^6 - 1) = .057\ldots.$$

By comparison, we observe that

$$\sum_{k=10^6}^{10^7} \frac{1}{k \ln k} < \ln\ln 10^7 - \ln\ln(10^6 - 1) = .154\ldots$$

and

$$\sum_{k=10^6}^{10^7} \frac{1}{k} < \ln 10^7 - \ln(10^6 - 1) = 2.303\ldots.$$