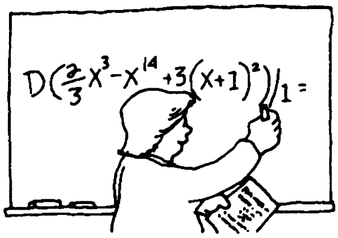
CLASSROOM CAPSULES

EDITOR

Warren Page
30 Amberson Ave.
Yonkers, NY 10705



Classroom Capsules consists primarily of short notes (1–3 pages) that convey new mathematical insights and effective teaching strategies for college mathematics instruction. Please submit manuscripts prepared according to the guidelines on the inside front cover to the Editor, Warren Page, 30 Amberson Ave., Yonkers, NY 10705-3613.

The Undying Novena

Christopher M. Rump, (crump@eng.buffalo.edu), SUNY Buffalo, Buffalo, NY 14260

A few years back, I received a chain letter from my sister that read

> This is not a chain letter, but an ongoing novena started in 1952. It is to St. Theresa the Little Flower. It has never been broken. Within 24 hours send 4 copies to friends and family and mention who sent it to you.

At first, I was highly suspicious of the stated birthdate of this letter and its claimed continuity. If no one ever broke the chain, I naively reasoned, then the exponential growth of mail would have overwhelmed the earth and stripped the world's forests for paper. (Today's electronic novena would similarly swamp the Internet with e-mail.) However, I soon realized that it is possible that the chain has never completely died out, although many individuals have broken their part of it.

Despite its claim, the letter is a type of chain letter, whose life can be modeled as a branching process [**2**]. In this stochastic process, each parent may give birth to four offspring creating four branches. Of course, some of the branches may die out as individuals break the chain. As long as one branch survives, however, the chain survives another generation. Since a novena requires nine consecutive days of prayer, we assume that a new generation occurs after every nine days—time sufficient for both the post office and the recipient to handle the letter before sending it onward.

To investigate the purported lifetime of the novena, let's assume that a person either breaks the chain, with probability p, or continues it as instructed, with probability $1 - p$. Then the mean number of offspring from each person is $m = 4(1 - p)$. Thus, each recipient of the novena sends out, on average, m novenas. Each of these m recipients sends out, on average, m novenas, and so on. Thus, in the nth generation there are, on average, $4m^n$ novena recipients stemming from the four original recipients in generation zero.

 © THE MATHEMATICAL ASSOCIATION OF AMERICA

For $m < 1$, the average generation size approaches zero. It turns out, in fact, that if $m \leq 1$ (i.e., $p \geq 3/4$) then the process will die out with probability one. In this case, the probability that the chain would have survived 45 years (over 1,800 generations) is infinitesimal. If we are to believe the letter's claimed longevity, we would have to infer that $p < 3/4$.

When $p < 3/4$ (i.e., $m > 1$) each person is replaced by more than one, on average. Thus, the process can exponentially explode as discussed in [**1**] and [**3**]. However, this is only the average case. The chain may still eventually die out and to find the probability that it does so, we seek the extinction probability π_0 of this process. The survival probability would then be $1 - \pi_0$.

Clearly, $\pi_0 = \pi^4$, where π is the extinction probability for a subtree rooted by one of the original four recipients. To solve for π we use a conditioning argument: an original recipient can kill the subtree immediately with probability p, or, with probability $1 - p$, will send out four copies of the novena, generating a new tree with extinction probability $\pi_0 = \pi^4$. Thus, π is a solution of $\pi = p + (1 - p)\pi^4$. Using a mathematical solver such as Maple or Mathematica, we find that $\pi = \frac{(3y-2)(3y+1)}{9y}$, where $y = x^{1/3}$ and

$$x = \frac{(7 + 20p) \pm 3\sqrt{3}\sqrt{3 + 8p + 16p^2}}{54(1 - p)}.$$

The extinction probabilities π and π_0 are plotted, as functions of the branching probability p, in Figure 1. We observe that the extinction probability π dominates p, as it must since $\pi = p + (1 - p)\pi^4$, but the two probabilities are very close when p is small. If $p = 0.5$, i.e., half the people break the chain, then $\pi \approx 0.54369$. The probability that the entire process eventually dies out, $\pi_0 = \pi^4$, is much smaller, however, as each of the four individual subtrees must die out. For $p = 0.5$, the novena dies out with probability $\pi_0 \approx 0.08738$. Hence, under this assumption, the chance that our novena could have survived this long exceeds 91%!

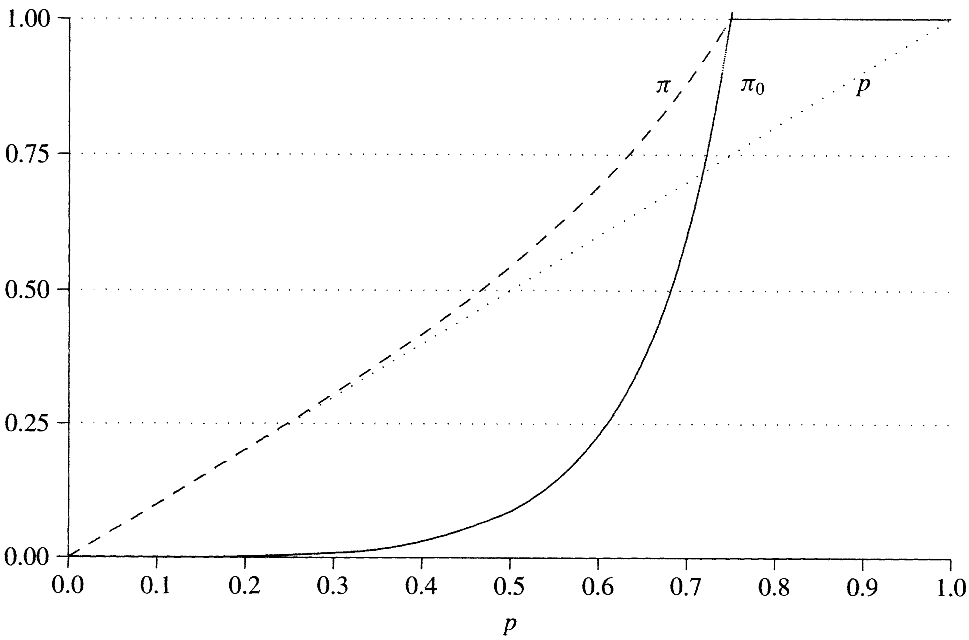


Figure 1. Extinction Probability Plot

Incidentally, in the end, I sent four copies of the letter to my sister and her husband, perhaps breaking the rules in spirit but not exactly breaking the chain. If she had sent 16 copies back to me we would have had quite a bit of exponential fun!

References

1. J. O. Case, The chain letter: an example of exponential growth, *Mathematics Teacher* **80** (1987), 114–115.
2. S. M. Ross, *Introduction to Probability Models*, 7th ed., Academic Press, San Diego, 2000.
3. D. J. Thuente, Chain letters: a poor investment unless..., *Two-Year College Mathematics Journal* **13** (1982), 28–35.

The Distance Between Two Graphs

Rhonda Huettenmueller (rkh0018@unt.edu), University of North Texas, Denton, TX 76203

Given the graphs of two functions—graphs that do not intersect—one might wonder about the minimum distance between them and how it is calculated. On first consideration, it would seem that a solution to this problem requires multivariable calculus because the distance between points on the two graphs is a function of their x-coordinates. But here is a way to solve some such problems in a beginning calculus course using single variable methods.

As an example, let us examine the graphs of $f(x) = e^x$ and $g(t) = -t^2$ (see the figure). It appears that the minimum distance between these graphs occurs along a line segment that is perpendicular to both graphs. If this is indeed the case, then we can find the values of x and t for which the line segment joining (x, e^x) and $(t, -t^2)$ is perpendicular to the tangent lines to the graphs at these two points. This idea is represented in the system

$$\frac{-1}{e^x} = \frac{e^x - (-t^2)}{x - t} = \frac{-1}{-2t}.$$

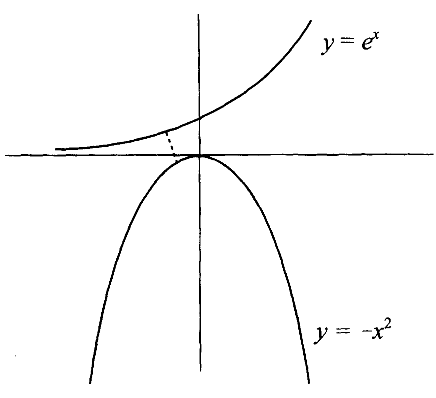


Figure 1.

 © THE MATHEMATICAL ASSOCIATION OF AMERICA