If a/b is very large, then $x_0 = w/\sqrt{(a/b)^2 - 1}$ is close to zero, so the incorrect intuition that f is minimized at 0 becomes correct "in the limit."

It is interesting to note that when $a > b$ the *maximum* of f occurs either at 0 or l: at 0 if $w \geq ((a^2 - b^2)/2ab)l$ and at l if $w \leq ((a^2 - b^2)/2ab)l$.

Taylor's Formula via Determinants

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For calculus students who know determinants one can, after doing Rolle's theorem, proceed to the following

Theorem. Let $f(x), f_1(x), \ldots, f_{n+2}(x)$ be $n+1$ times continuously differentiable functions. Then

$$\begin{vmatrix} f(x) & f_1(x) & \ldots & f_{n+2}(x) \\ f(0) & f_1(0) & \ldots & f_{n+2}(0) \\ f'(0) & f_1'(0) & & f_{n+2}'(0) \\ & \ldots & & \\ f^{(n)}(0) & f_1^{(n)}(0) & \ldots & f_{n+2}^{(n)}(0) \\ f^{(n+1)}(h) & f_1^{(n+1)}(h) & \ldots & f_{n+2}^{(n+1)}(h) \end{vmatrix} = 0 \qquad (1)$$

for some h between 0 and x.

Proof. Consider x as constant and let $D^{(i)}(h)$ denote the function of h obtained by replacing the last row of the determinant with $f^{(i)}(h)\, f_1^{(i)}(h) \ldots f_{n+2}^{(i)}(h)$. Observe that for $i = 0, 1, \ldots, n$ the derivative of $D^{(i)}(h)$ with respect to h is $D^{(i+1)}(h)$ and the determinant in (1) is $D^{(n+1)}(h)$. Now $D^{(0)}(0) = 0$ because the second and the last rows are the same; likewise, $D^{(0)}(x) = 0$ because its first and last rows are the same. So, by Rolle's theorem, $D^{(1)}(h) = 0$ for some h between 0 and x. Also, the last row of $D^{(1)}(0)$ is the same as its third. So, using Rolle's theorem again, $D^{(2)}(h) = 0$ for some h between 0 and x. Continuing, we see that $D^{(n+1)}(h) = 0$ for a suitable h between 0 and x. *q.e.d.*

For example, (1) shows that for some h between 0 and x, we have

$$\begin{vmatrix} f(x) & 1 & \frac{x}{1!} & \frac{x^2}{2!} & \ldots & \frac{x^n}{n!} & \frac{x^{n+1}}{(n+1)!} \\ f(0) & 1 & 0 & 0 & \ldots & 0 & 0 \\ f'(0) & 0 & 1 & 0 & \ldots & 0 & 0 \\ & & & \ldots & & & \\ f^{(n)}(0) & 0 & 0 & 0 & \ldots & 1 & 0 \\ f^{(n+1)}(h) & 0 & 0 & 0 & \ldots & 0 & 1 \end{vmatrix} = 0$$

which is Taylor's formula because the determinant is

$$f(x) - f(0) - \frac{x}{1!}f'(0) - \frac{x^2}{2!}f''(0) - \cdots - \frac{x^n}{n!}f^{(n)}(0) - \frac{x^{n+1}}{(n+1)!}f^{n+1}(h).$$

