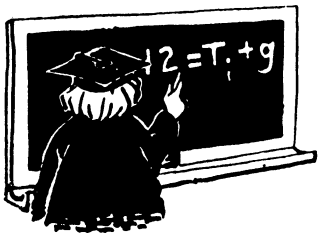

CLASSROOM CAPSULES

Edited by
Frank Flanigan

A Classroom Capsule is a short article that contains a new insight on a topic taught in the earlier years of undergraduate mathematics.

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The AM-GM Inequality via $x^{1/x}$

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Using elementary properties of the derivative it is a simple matter to show that if $x > 0$, then $f(x) = x^{1/x}$ has an absolute maximum at $x = e$. Thus if x is any positive real number, then $e^{1/e} \geq x^{1/x}$ or

$$e^x \geq x^e \tag{1}$$

with equality if and only if $x = e$. Inequality (1) provides an easy way of proving the inequality between the arithmetic and geometric means.

Let $a_1, a_2, \ldots, a_n$ be any n positive real numbers with arithmetic mean A and geometric mean G. To prove $A \geq G$, we substitute $x = a_1 e/G$, $x = a_2 e/G, \ldots,$ $x = a_n e/G$, in turn into (1), and multiply the resulting inequalities. Hence

$$e^{(a_1 e/G + a_2 e/G + \cdots + a_n e/G)} \geq \left(\frac{a_1 e}{G} \cdot \frac{a_2 e}{G} \cdots \frac{a_n e}{G} \right)^e. \tag{2}$$

Since $a_1 + a_2 + \cdots + a_n = nA$ and $a_1 a_2 \cdots a_n = G^n$, it follows from (2) that

$$e^{nAe/G} \geq \left(\frac{e^n G^n}{G^n} \right)^e = e^{ne}.$$

Thus $nAe/G \geq ne$ or $A \geq G$. Furthermore $A = G$ if and only if each of the substituted values of x is e. Hence

$$\frac{a_1 e}{G} = e, \frac{a_2 e}{G} = e, \ldots, \frac{a_n e}{G} = e$$

from which it follows that $a_1 = a_2 = \cdots = a_n$.