forms for a given answer. (Any large group of test-takers will discover at least a few that hadn't occurred to us.) There is essentially *only one* correct 4 SD answer—the infinity of variations beyond that number of places don't have to be checked! The choice of "4" is arbitrary, but it is large enough to eliminate guessing, and it is small enough to override differences among calculators or (in most cases) disastrous cancellations.

Of course, we usually cannot tell how a student arrived at a wrong decimal approximation. (Since it is in the nature of calculus problems that the student's resort to the calculator comes late in the computation, an incorrect symbolic result is also probably close at hand.) However, it is possible that a somewhat less accurate decimal approximation may result from greater understanding of mathematics than that being tested. For example, a student required to calculate a definite integral, but who cannot remember the appropriate antiderivative, might resort to the trapezoidal rule, with enough steps to get 2 or 3 SD. This approach should be rewarded, not penalized, since it demonstrates a better understanding of integration than most of our students ever acquire.

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Finding Rational Roots of Polynomials

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In "Synthetic Division Shortened" [*TYCMJ* 12 (November 1981) 334–336], Warren Page and Leo Chosid gave a very useful necessary condition for a polynomial with integral coefficients to have a rational root. In this capsule, we provide two additional results designed to ease the work involved in finding rational roots of polynomials with integer coefficients. Although both of these results are known, neither seems to be readily available in the literature. The proofs given here are quite simple.

Let us begin by stating the rational root theorem.

Theorem 1. *Let $f(x)=a_nx^n+a_{n-1}x^{n-1}+\cdots+a_1x+a_0$ be a polynomial all of whose coefficients are integers. If $f(p/q)=0$ for relatively prime integers p and q, then $p|a_0$ and $q|a_n$.*

The procedure for finding the rational roots of the polynomial $f(x)$ is to list all possible rational numbers p/q such that $p|a_0$ and $q|a_n$, and to see which, if any, satisfy $f(p/q)=0$. Of course, this task isn't quite as arduous as it looks, since we can use Descartes' rule of signs and results on upper and lower bounds for the zeros to eliminate the need to check every possibility. And those rationals that need to be tested can be checked rather quickly by the Page-Chosid method alluded to above. However, if a_0 and a_n have many factors, there could still be many rational numbers to check.

Our first result says that if certain conditions are fulfilled, then the polynomial has no rational roots.

Theorem 2. *Let $f(x)$ be a polynomial of degree at least two defined as in Theorem 1. If a_0, a_n and $f(1)$ are all odd, then $f(x)$ has no rational roots.*

Proof. The proof is by contradiction using a simple parity argument. Suppose p/q is a rational root of $f(x)$, where p and q are relatively prime. Then (Theorem 1) $p|a_0$ and $q|a_n$, and (since a_0 and a_n are both odd integers) both p and q are odd integers. Thus, for every pair of nonnegative integers satisfying $k+j>0$, we have

$$p^k q^j - 1 \equiv 0 \pmod{2}. \tag{1}$$

From $f(p/q)=0$, we have

$$q^n f(p/q) = a_n p^n + a_{n-1} p^{n-1} q + \cdots + a_0 q^n = 0.$$

Since $f(1)$ is also odd, we have

$$f(1) - q^n f(p/q) = f(1) \equiv 1 \pmod{2}. \tag{2}$$

But, by (1),

$$\begin{aligned} f(1) - q^n f(p/q) &= a_n(1-p^n) + a_{n-1}(1-p^{n-1}q) + \cdots + a_0(1-q^n) \\ &\equiv 0 \pmod{2}. \end{aligned} \tag{3}$$

The contradictory results (2), (3) show that p/q cannot be a root of $f(x)$. Thus, the polynomial has no rational roots.

Example. The polynomial

$$x^5 + 7x^4 - 28x^3 + 125x^2 + x - 3275$$

has no rational roots since a_5, a_0, and $f(1) = -3169$ are all odd. Thus, we need not contemplate what to do with the 16 divisors of -3275.

Our second result is the following.

Theorem 3. *Let $f(x)$ be as in Theorem* 1 *and let a be an integer such that $f(a) \neq 0$. If p and q are relatively prime integers such that $f(p/q)=0$, then*

$$(p-aq)|f(a).$$

Proof. We begin by noting that $f(x)-f(a)$ is a polynomial with integer coefficients that has a as a root. Thus, by the factor theorem, there is a polynomial $g(x)$ with integer coefficients such that $(x-a)g(x) = f(x) - f(a)$. Therefore,

$$\left(\frac{p}{q} - a\right) g(p/q) = f(p/q) - f(a) = -f(a). \tag{4}$$

If the polynomial $f(x)$ has degree n, then $g(x)$ has degree $n-1$, and so $q^{n-1}g(p/q)$ is an integer. Since p and q are relatively prime, $p-aq$ and q are relatively prime. Multiplying both sides of (4) by q^n we obtain

$$q^{n-1}(p-aq)g(p/q) = -q^n f(a). \tag{5}$$

Since $q^{n-1}g(p/q)$ is an integer, $p-aq$ divides the right-hand side of (5). Since $p-aq$ and q are relatively prime, $(p-aq)|f(a)$, as asserted in our theorem.

Remark. If $f(a)=0$, then $(p-aq)|f(a)$ for any p and q. If we take $a=0$, then $f(a)=a_0$ and we get $p|f(0)=a_0$, which is part of the statement of the rational root theorem (Theorem 1). This is why we have considered only nonzero a.

Example. Using Descartes' rule of signs we see that

$$f(x) = 36x^6 - 96x^5 + 49x^4 + 47x^3 - 33x^2 - 7x + 4$$

has four, two, or zero positive roots, and two or zero negative roots. To apply Theorem 3, we would like to find an $a \neq 0$ so that $f(a) \neq 0$. We try $a = 1$ and find that $f(1) = 0$. We have found a root of $f(x)$, but not an a to which we can apply Theorem 3. Testing $a = -1$ yields $f(-1) = 112$. Thus, a rational root p/q of $f(x)$ must satisfy $(p+q)|112$. A particular solution is $p = 1$ and $q = 1$.

Theorem 1 tells us that all rational roots of $f(x)$ must lie among the 30 rational numbers

$$1, -1, 2, -2, 4, -4, 1/2, -1/2, 1/3, -1/3, 2/3, -2/3, 4/3,$$
$$-4/3, 1/4, -1/4, 1/6, -1/6, 1/9, -1/9, 2/9, -2/9, 4/9, -4/9,$$
$$1/12, -1/12, 1/18, -1/18, 1/36, -1/36.$$

Based on Theorem 3, we know that only the following eleven of these rational numbers may be possible roots:

$$1, -2, -1/2, 1/3, -1/3, -2/3, 4/3, -4/3, 1/6, -1/9, -2/9.$$

Since $f(1) = 0$, we begin our synthetic division and obtain

1	36	−96	49	47	−33	−7	4
		36	−60	−11	36	3	−4
1	36	−60	−11	36	3	−4	0
		36	−24	−35	1	4	
	36	−24	−35	1	4	0	

Thus, 1 is a double root. One can also readily verify that $-1/2$ is a double root, and $4/3$ and $1/3$ are simple roots.

It is also possible to use several different values of a to eliminate more rational numbers from the list of potential rational roots.

Example. To find the rational roots of

$$f(x) = 60x^6 - 212x^5 + 203x^4 + 48x^3 - 133x^2 + 10x + 24,$$

we first look for divisors of 60 and 24. This gives us (after deleting the repeats) 72 possible rational roots, which we will not list. Since $f(1) = 0$, we know that $a = 1$ is a root. Since $f(-1) = 308$ and $f(2) = 200$, we know (Theorem 3) that a rational root p/q must satisfy

$$(p+q)|308 \quad \text{and} \quad (p-2q)|200.$$

These constraints reduce our list of 72 possible rational roots to the following 16 possibilities:

$$1, -2, 3, -3, 6, -8, -1/2, 1/3, -2/3, 4/3, -4/3, 8/3, 3/4, 2/5, 6/5, -1/12.$$

Readers can verify that $f(x)$ has 1 and $-1/2$ as double roots, and $4/3$ and $6/5$ as simple roots.

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Another Proof of Chebysheff's Inequality

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If x_i and y_i $(i = 1, 2, \ldots, n)$ are real numbers such that

$$x_1 \geq x_2 \geq \cdots \geq x_k \geq \cdots \geq x_n \quad \text{and} \quad y_1 \geq y_2 \geq \cdots \geq y_k \geq \cdots \geq y_n, \qquad (1)$$