Now consider the 3×3 square with opposite vertices at $(a+1, b+1)$ and $(a+3, b+3)$. None of the points in the square is visible from the origin. Since $a \equiv -r \pmod{a_r}$ and $b \equiv -s \pmod{b_s}$ for $1 \leq r \leq 3$, $1 \leq s \leq 3$, the prime in row r and column s of M divides both $a+r$ and $b+s$. Hence $a+r$ and $b+s$ are not relatively prime, and thus the lattice point $(a+r, b+s)$ is not visible from the origin.

This particular 3×3 square of invisible lattice points is far from the origin. Perhaps we can find another 3×3 square of invisible lattice points that is closer by selecting different primes, or the same primes in a different order, for the entries of M.

If we use the first sixteen primes to construct a 4×4 square of invisible lattice points, we would have to make calculations modulo $2 \times 3 \times \cdots \times 53 = 32589158477190044730$. Such a project is beyond the courage of this author!

Exploring the Volume-Surface Area Relationship

Keith A. Struss (student), Rose-Hulman Institute of Technology, Terre Haute, IN

At first glance it appears to be a coincidence that the surface area of a sphere may be found by taking the derivative of its volume function. However, as we shall see, a particular mathematical relationship often holds between the volume of an object and its surface area—namely,

$$dV = A\,d\tau, \tag{1}$$

where dV is the increase in volume of the solid that would result from coating it with a uniform layer of thickness $d\tau$, and A is the surface area of the solid. Of course, dV and $d\tau$ are infinitesimals. To see that this is reasonable, recall from elementary calculus that the volume of some solids may be computed by using cross-sectional areas as follows

$$V(x) = \int_a^x A(u)\,du. \tag{2}$$

Now, let $A(\tau)$ denote the surface area of the solid when it is uniformly coated by a coating of thickness τ. A natural analogue of (2) is

$$V(\tau) = \int_0^\tau A(u)\,du \tag{3}$$

where $V(\tau)$ is the additional volume arising from coating the solid; therefore differentiating and evaluating the expression at $\tau = 0$ yields

$$\left.\frac{dV}{d\tau}\right|_{\tau=0} = A(0), \tag{4}$$

which is just another way of expressing (1).

Let's verify this for the sphere. Since $V = (4/3)\pi r^3$, we know that

$$dV = \frac{dV}{dr}dr = 4\pi r^2 dr. \tag{5}$$

Now imagine that a coating of thickness $d\tau$ is painted on the solid. Then the radius increases by $d\tau$. In other words, $dr = d\tau$ in (5). Hence, (1) holds.

Let us now consider (1) for a rectangular box having dimensions l, w, and h. Since the box has volume lwh, the total differential is

$$dV = (lw)\,dh + (lh)\,dw + (wh)\,dl. \tag{6}$$

But if the box is covered by a uniform coat of paint of thickness $d\tau$, then $dl = dw = dh = 2\,d\tau$. Therefore, $dV = 2lw\,d\tau + 2lh\,d\tau + 2wh\,d\tau = A\,d\tau$.

Another solid that we can easily test is the right circular cylinder. The cylinder of radius r and height h has volume $V = \pi r^2 h$. The total differential of V is

$$dV = \pi r^2\,dh + 2\pi rh\,dr. \tag{7}$$

If we imagine that the entire surface of the cylinder is painted over by a layer of paint that has thickness $d\tau$, then the radius would increase by amount $d\tau$ and the height would increase by amount $2\,d\tau$. Substituting $dr = d\tau$ and $dh = 2\,d\tau$ into (7), we obtain

$$dV = 2\pi r^2\,d\tau + 2\pi rh\,d\tau = (2\pi r^2 + 2\pi rh)\,d\tau. \tag{8}$$

Thus, (1) and (8) yield $A = 2\pi r^2 + 2\pi rh$.

Our next geometric result is for a cone, which has volume $V = (1/3)\pi r^2 h$. We will derive the formula for the lateral surface area. Taking the differential produces

$$dV = \frac{1}{3}\pi r^2\,dh + \frac{2}{3}\pi rh\,dr. \tag{9}$$

There is one "sticky" point that must now be addressed. If the cone is truly coated all over with an equal thickness of paint, then the new solid is *no longer a cone!* In particular, dV does not include the increased volume $\pi r^2 d\tau$ of the new solid's cylindrical base. However, since dr and dh are small, we can find the lateral surface area of the cone by coating its lateral surface area and comparing this with the larger cone of radius $r + dr$ and height $h + dh$. (See Figure 1.)

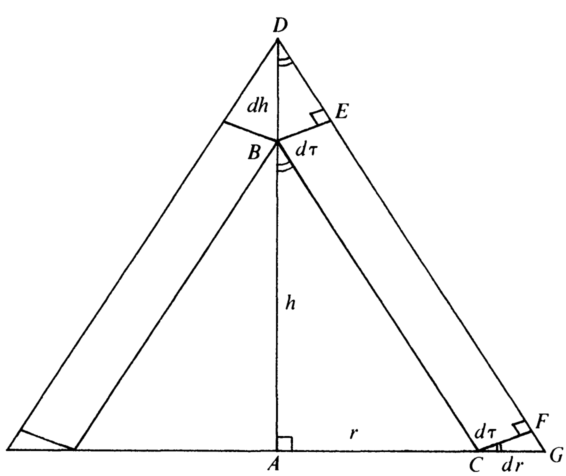


Figure 1

To find dh and dr in terms of $d\tau$, use the fact that triangles ABC, CGF, and DEB are similar. Since triangle ABC is similar to triangle BDE, their sides are in the ratio $dh/d\tau = \sqrt{h^2 + r^2}/r$, and so

$$dh = \frac{\sqrt{h^2 + r^2}}{r} d\tau.$$

Since triangles ABC and CFG are similar, their sides are in the ratio $dr/d\tau = \sqrt{h^2 + r^2}/h$, and therefore

$$dr = \frac{\sqrt{h^2 + r^2}}{h} d\tau.$$

Hence, (9) can be written as

$$dV = \left[\frac{1}{3}\pi r^2 \frac{\sqrt{r^2 + h^2}}{r} + \frac{2}{3}\pi rh \frac{\sqrt{r^2 + h^2}}{h} \right] d\tau = \pi r \sqrt{r^2 + h^2}\, d\tau. \tag{10}$$

Since $s = \sqrt{r^2 + h^2}$ is the slant height of the cone, we see that (10) and (1) yield the lateral surface area $A = \pi rs$.

Now, let us derive a formula for the lateral surface area of the solid of revolution for a function $f(x)$. We define the function $f(x, \tau)$ which yields the height of our original function $f(x)$ plus the additional height caused by "coating" $f(x)$ with a uniform coating of thickness τ. (See Figure 2.) Note that, in general, $f(x, \tau) \neq f(x) + \tau$. Also note that $f(x,0) = f(x)$. Our "coated" surface of revolution will be obtained by revolving the function $y = f(x, \tau)$ between $x = a$ and $x = b$ about the x-axis. If the solid is painted uniformly with an infinitesimal thickness $d\tau$, then this

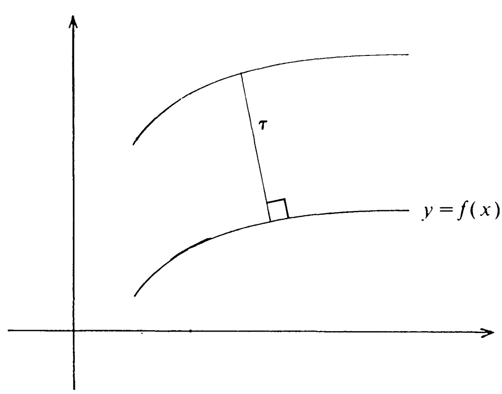


Figure 2

will cause an increase df in the height of the function (Figure 3). To find the relationship between df and $d\tau$, observe that $\tan\theta = f'(x)$. Note also that $df = d\tau \sec\theta$. Since $1 + \tan^2\theta = \sec^2\theta$, we take the partial derivative with respect to τ and evaluate at $\tau = 0$, as done in (4), thus obtaining $df = \sqrt{1 + (f'(x))^2}\, d\tau$, or

$$\left. \frac{\partial f(x, \tau)}{d\tau} \right|_{\tau=0} = \sqrt{1 + (f'(x))^2}. \tag{11}$$

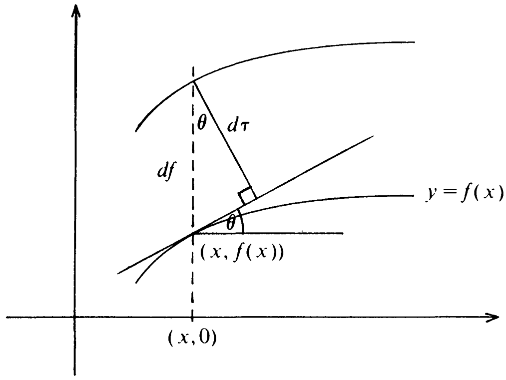


Figure 3

Beginning with the formula for the volume of the corresponding solid of revolution

$$V = \int_a^b \pi [f(x,\tau)]^2 \, dx, \tag{12}$$

taking the derivative with respect to τ and evaluating at $\tau = 0$ yields

$$\begin{aligned} \left.\frac{dV}{d\tau}\right|_{\tau=0} &= \int_a^b 2\pi f(x,\tau) \left.\frac{\partial f(x,\tau)}{\partial \tau}\right|_{\tau=0} dx \\ &= \int_a^b 2\pi f(x) \sqrt{1 + (f'(x))^2} \, dx. \end{aligned} \tag{13}$$

Thus, (13) and (4) yield the well-known formula

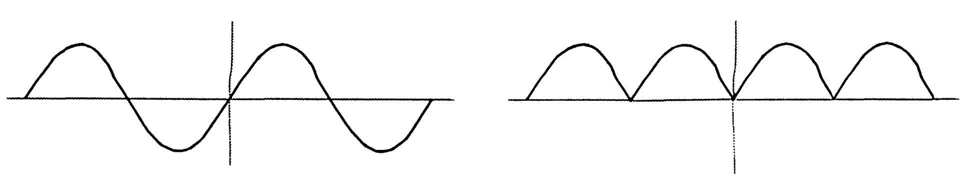
$$\text{Lateral Surface Area} = \int_a^b 2\pi f(x) \sqrt{1 + (f'(x))^2} \, dx$$

for a surface of revolution.

$\mathrm{Sin}^2 x$: A Sheep in Wolf's Clothing

Mark E. Saul, Bronxville Schools, Bronxville, NY

Students occasionally must construct the graph of $y = \sin^2 x$. Noting that both squaring and taking absolute values produce positive numbers as results, they frequently construct the graph of $y = |\sin x|$ instead, by taking the elementary sine curve and "flipping it up" to get positive values only:



Since this procedure gives an accurate representation of the periodicity of the