Moreover, the quantity dV approximates ΔV (the exact change in volume) when Δr and Δh are small.

Suppose that the cylinder is a standard beer can with dimensions as given above. Then we have

$$\Delta V \approx dV = \pi(10\Delta r + \Delta h).$$

This result expresses the fact that, at these dimensions, the volume is approximately 10 times more sensitive to changes in radius than to changes in height. Hence one can make a beer can appear to be larger than the standard one by decreasing the radius slightly (so little as to be hardly noticeable) and increasing the height so no change in volume results. The sensitivity analysis above shows that even a tiny decrease in radius forces an appreciable compensating increase in height.

Smart people, those marketing and sales types.

An Optimization Oddity

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Suppose f is a differentiable function ($f(x) \geq 0$) on the interval $[0, t]$ ($t > 0$) such that d^2f/dx^2 exists, is different from zero, and does not change sign on the same interval. In addition, without loss of generality, we assume $d^2f/dx^2 < 0$ for all $x \in [0, t]$. Consider a tangent m to f at the point $(a, f(a))$, $a \in [0, t]$. The problem is to find the place $x = a$ such that the tangent m minimizes the area $\Phi(a)$ of the region determined by $m, f, x = 0$, $x = t$. (See Figure 1.) Surprisingly, the result is independent of f!

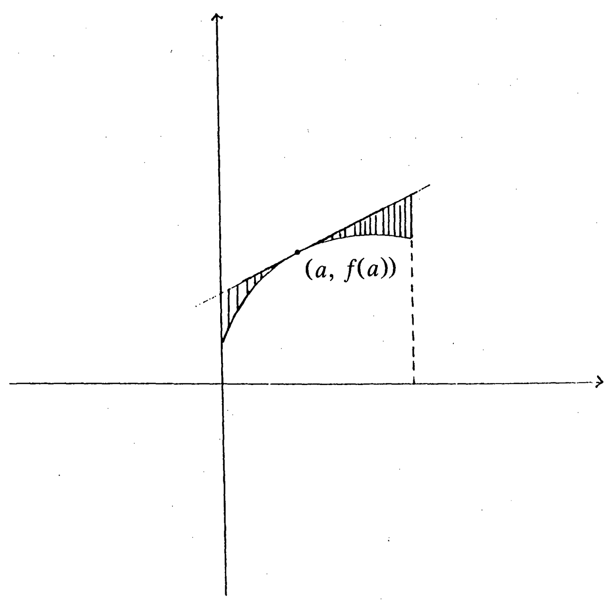

Figure 1

First, let us consider a calculus proof. Now

$$\Phi(a) = \int_0^t f'(a)(x-a) + f(a) - f(x)\,dx.$$

Since the limits of integration are independent of a, we differentiate with respect to a under the integral sign [2, page 380] to obtain

$$\Phi'(a) = \int_0^t f''(a)(x-a)\,dx. \tag{1}$$

Integrating and setting equal to 0, we have

$$tf''(a)(t-2a) = 0 \Rightarrow a = \frac{t}{2}.$$

Thus, if the area function Φ has a local extreme value it must be taken at the place $a = t/2$. To decide the behavior of Φ at $a = t/2$ we examine the second derivative of Φ. Differentiating both sides of (1), again with respect to a, we obtain

$$\begin{aligned}\Phi''(a) &= \int_0^t f'''(a)(x-a) - f''(a)\,dx \\ &= f'''(a)\left(\frac{t^2}{2} - at\right) - f''(a)t.\end{aligned}$$

At $a = t/2$ we obtain

$$\Phi''\left(\frac{t}{2}\right) = -f''\left(\frac{t}{2}\right)t > 0 \quad \text{by hypothesis.}$$

Thus, the function Φ does indeed have a minimum at $a = t/2$ that is completely independent of f.

This problem also fits quite nicely in the area of convex geometry, from which we draw the following proof. For convenience, and without loss of generality, we restate our result as follows:

> Let f: $[0,t] \to \mathbb{R}$ be a concave function with only positive values (which implies the continuity of f on the open interval $(0,t)$ [2, p. 199], thus guaranteeing the integrability of f), and let g: $[0,t] \to \mathbb{R}$ be a linear function with $g \geq f$. If $\int_0^t g(x) - f(x)\,dx$ is minimal with respect to the set of all linear functions $\tilde{g} \geq f$ then $g(t/2) = f(t/2)$.

Proof. Clearly, $f(t/2) \leq g(t/2)$. Take $A_f = \int_0^t f(x)\,dx$. The area A_g of the trapezoid with the vertices $0, (t,0), (t, g(t)), (0, g(0))$ is $t \cdot g(t/2)$. By hypothesis the difference $\Phi = A_g - A_f$ is minimal.

The region D determined by the graph of f, $x = 0$, $y = 0$, $x = t$ is convex. Take a supporting line, [1], containing the point $(t/2, f(t/2))$ and let h: $[0,t] \to \mathbb{R}$ be the restriction of the corresponding linear function. Then we also have $f \leq h$. Forming the trapezoid with h instead of g we obtain as area $A_h = t \cdot f(t/2)$. From the mentioned minimality we obtain $A_g \leq A_h$ and so $g(t/2) \leq f(t/2)$. ∎

In the statement given for this point of view we have made no claim about the existence of a minimizing linear function g. But from the proof it is clear that any supporting line of D containing the point $(t/2, f(t/2))$ gives rise to a suitable function. In contrast to the differentiable case there may be more minimizing functions; to be precise, note that the one-sided derivatives $f'_+(t/2)$ and $f'_-(t/2)$ exist with $f'_+(t/2) \le f'_-(t/2)$ [2, p. 199, theorem 4.43] implying that all linear functions

$$g(x) = m\left(x - \frac{t}{2}\right) + f\left(\frac{t}{2}\right)$$

with slope m such that $f'_+(t/2) \le m \le f'_-(t/2)$ are all supporting lines of D through $(t/2, f(t/2))$ and hence minimize Φ [2, p. 200, theorem 4.44].

Although the property described looks very elementary we have not been able to locate something similar in the literature.

References

1. James R. Smart, *Modern Geometries*, 3rd ed., Brooks/Cole, Pacific Grove, CA, 1988, p. 91.
2. Karl R. Stromberg, *Introduction to Classical Real Analysis*, Wadsworth, Belmont, CA, 1981.

A Simple Geometric Proof of the Addition Formula for the Sine

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The following is a short, simple, but not well-known proof of the well-known trignometric identity $\sin(a+b) = \sin a \cos b + \cos a \sin b$. The proof can be easily understood by students and uses nothing more complicated than the (triangle-based) definitions of sine and cosine and the area formula for triangles, $A = \frac{1}{2}$(base)(height). In the figure, a, b, and $a+b$ are all acute angles. With appropriate minor modifications, a similar argument applies if b and/or $a+b$ are obtuse.

From the area formula, we see that the product of base and height is independent of which side of the triangle is chosen as the base. So in triangle PQR, we have $\mathrm{QR} \cdot \mathrm{PS} = \mathrm{QP} \cdot \mathrm{RT}$ and thus

$$\mathrm{PS} = \frac{\mathrm{PQ} \cdot \mathrm{RT}}{\mathrm{QR}} = \frac{(\mathrm{QT} + \mathrm{TP})\mathrm{RT}}{\mathrm{QR}}. \qquad (*)$$