After hearing a recording of this music, students enjoy analyzing the piece, finding that the esthetic appeal and intellectual attractiveness of "Clapping Music" can, in part, by explained by three factors: the complexity of pattern allowed by 12 beats composed of four pauses and eight claps, the variations that result from the application of such a simple cyclic permutation, and the syncopation provided by the particular 3, 2, 1, 2 pattern used by Reich. Students find these considerations far more exciting than counting beaded necklaces! The use of materials from the humanities in mathematics classrooms can be invaluable in maintaining students' interest.

'Hidden' Boundaries in Constrained Max-Min Problems

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In my first period Calc III class we considered the problem of finding the minimum distance from the origin to the paraboloid $z = 4 - x^2 - 4y^2$. The first octant of the constraint surface is shown in Figure 1.

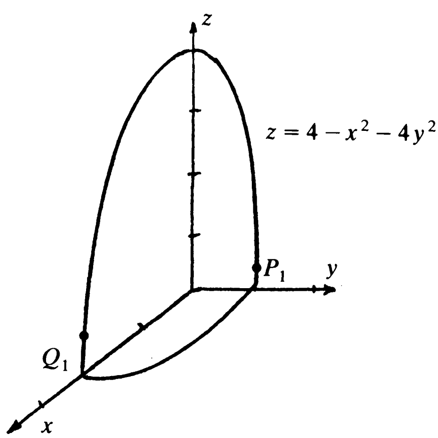

Figure 1

Minimizing distance-squared and replacing x^2 by $4 - 4y^2 - z$ gives the following function of two variables to be minimized

$$D(y, z) = 4 - 4y^2 - z + y^2 + z^2 = 4 - 3y^2 + z^2 - z.$$

Setting the partial derivatives D_y and D_z to zero gives

$$D_y = -6y \quad = 0$$
$$D_z = 2z - 1 = 0.$$

Thus the only critical point of D is $y = 0$, $z = 1/2$. Solving for x on the paraboloid gives the points $Q_1(\sqrt{7/2}, 0, \frac{1}{2})$ and $Q_2(-\sqrt{7/2}, 0, \frac{1}{2})$. Q_1 is shown in Figure 1.

It was clear to the students that the distance from the origin to Q_1 or Q_2 is not minimal since the point $(0, 1, 0)$ is closer. I was at a loss to explain why the critical points did not include the expected global minimum, but I was saved when the first period ended.

In the second period class we tried the same problem but used the constraint equation to eliminate y^2 instead of x^2. This time the resulting critical points gave the expected points $P_1(0, \sqrt{31/32}, 1/8)$ and $P_2(0, -\sqrt{31/32}, 1/8)$ corresponding to the global minimum. P_1 is shown in Figure 1.

This example seemed to show that we must consider all possible eliminations of variables via the constraint equation to be sure of finding the global extrema. This could be bad news! To resolve this apparent difficulty, I first went to my favorite calculus textbooks. The only hint of trouble that I found was in an example used by Thomas and Finney [**1**] to motivate Lagrange multipliers. I next consulted my busy colleagues and found them too busy with their own problems. When all else failed, I tried a similar problem in the plane and discovered that there was difficulty at a boundary. The problem in the plane was encountered in the paper "Exceptional extremum problems" [**2**] and was resolved in the paper "So-called exceptional extremum problems" [**3**]. (See also FFF 34, CMJ, March 1991.)

The operative theorem for global extrema problems states that if $f(x, y)$ is continuous on a bounded region R then its global extrema must be on the boundary of R or in the interior of R where $f_x = f_y = 0$ or in the interior where f_x or f_y fail to exist. For our problem let us cut off the unbounded part of the domain by requiring $z \geq -2$. This restriction will not exclude any minimum points since $x^2 + y^2 + z^2 \geq 4$ in the excluded region and we already know that the minimum is no bigger than one (the distance-squared from $(0,0,0)$ to $(0,1,0)$).

We now return to the first period substitution $x^2 = 4 - y^2 - z$ and note that since $x^2 \geq 0$ then $4 - 4y^2 - z \geq 0$. The boundary $z = 4 - 4y^2$ is the 'hidden' boundary that was not considered in the first hour solution. Thus we seek the global minimum of $D(y, z)$ on the bounded region $-2 \leq z \leq 4 - 4y^2$ shown as the cross-hatched region in Figure 2.

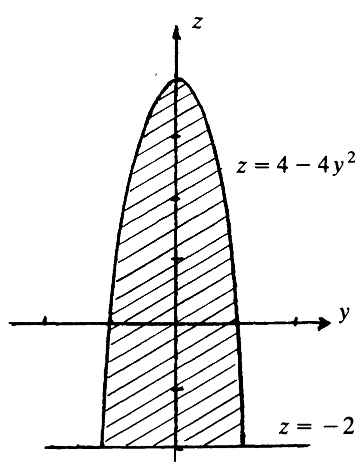

Figure 2

The interior candidates Q_1 and Q_2 have already been found. There are no boundary candidates on $z = -2$ since $D \geq 4$ on this boundary. Boundary candidates on the parabola $z = 4 - 4y^2$ are found by considering

$$E(y) = D(y, 4 - 4y^2) = 16y^4 - 31y^2 + 16, \qquad -\sqrt{3/2} \leq y \leq \sqrt{3/2}.$$

The minimal value of $E(y)$ must be either at its boundary points $y = \pm\sqrt{3/2}$ or at $y = 0$, $\pm\sqrt{31/32}$ where $dE/dy = 0$. Comparing $E(y)$ at these five values of y, we find the minimum at $y = \pm\sqrt{31/32}$. Solving for the corresponding z on the parabola and x on the paraboloid gives $P_1(0, \sqrt{31/32}, 1/8)$ and $P_2(0, -\sqrt{31/32}, 1/8)$ as the boundary candidates for minimal $D(y, z)$.

The global minimum for distance-squared is then found by comparing the values of $D(y, z)$ at Q_1, Q_2, P_1 and P_2. We find $D = \sqrt{13}/2$ at Q_1 and Q_2 and $D = \sqrt{63}/8$ at P_1 and P_2. Thus the global minimum is at P_1 and P_2 on the 'hidden' boundary $z = 4 - 4y^2$.

The method of Lagrange multipliers gives the same four candidates and also the point $(0, 0, 4)$ which we found using one of the values for which $dE/dy = 0$. In some cases the Lagrange method is algebraically easier but in many cases the students (and teachers) miss some of the candidates when solving the system of algebraic (often nonlinear) equations.

References

1. G. B. Thomas and R. L. Finney, *Calculus and Analytic Geometry*, 6th ed., Addison-Wesley, 1984, pp. 866–867.
2. C. S. Ogilvy, Exceptional extremum problems, *American Mathematical Monthly* 67 (1960) 270–275; reprinted in *Selected Papers on Calculus*, MAA, 1969, pp. 262–267.
3. H. A. Thurston, So-called exceptional extremum problems, *American Mathematical Monthly* 68 (1961) 650–652; reprinted in *Selected Papers on Calculus*, MAA, 1969, pp. 268–270.

Another Proof of a Familiar Inequality

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Let $y_1, y_2, \ldots, y_n$ be any permutation of the positive numbers $x_1, x_2, \ldots, x_n$. We use a simple inequality involving the logarithmic function to obtain the following familiar proposition:

$$x_1^{x_1} x_2^{x_2} \cdots x_n^{x_n} \geq y_1^{x_1} y_2^{x_2} \cdots y_n^{x_n} \tag{1}$$

with equality if and only if $x_i = y_i$ $(i = 1, 2, \ldots, n)$.

The standard proof of (1) uses a not particularly simple induction argument. [See D. S. Mitrinović, *Analytic Inequalities*, Springer-Verlag, NY, 1970, p. 284.]

If $x > 0$, then

$$x - 1 \geq \ln x, \tag{2}$$

with equality if and only if $x = 1$.

(2) follows immediately from the observation that $f(x) = \ln x - x + 1$ has an absolute maximum at $x = 1$, because $f'(x) = 1/x - 1$ vanishes if and only if $x = 1$, and $f''(x) = -1/x^2$ is negative for all x.

Substituting $x = y_i/x_i$ in (2) gives

$$\frac{y_i}{x_i} - 1 \geq \ln\frac{y_i}{x_i} \qquad (i = 1, 2, \ldots, n). \tag{3}$$