Area and Perimeter, Volume and Surface Area

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Most calculus teachers have observed that the derivative of the area of a circle with respect to its radius is the perimeter: $d/dr(\pi r^2) = 2\pi r$. Similarly, $d/dr(\frac{4}{3}\pi r^3) = 4\pi r^2$; the derivative of the volume of a sphere is the surface area. Jay Miller [**2**] showed that the derivative of the area of a regular polygon with respect to its apothem (the perpendicular distance from the center to any edge) is the perimeter, and he proved the analogous relation between the volume and surface area of a regular polyhedron. Wondering whether these results can be generalized, I was surprised to discover that in fact they are special cases of quite general relationships between the area and perimeter of a plane region, and between the volume and surface area of a solid.

The key is just the familiar observation that under a dilation $\mathbf{x} \mapsto m\mathbf{x}$, linear dimensions such as the perimeter of a region are magnified by m, the magnification factor of the dilation, but the area is magnified by the square of this factor. Thus if s is a characteristic linear dimension of a region, say a diameter or an edge length, then the perimeter L is related to s by a formula of the type $L = ks$ for some constant k, and the area is given by a formula of the type $A = cs^2$ for a constant c.

Example. For squares the side length s is a natural choice of characteristic length, in terms of which the perimeter and area are given by $L = 4s$ and $A = s^2$. Note that $dA/ds = 2s \neq L$. However, if we set $a = \frac{1}{2}s$ then $L = 8a$, $A = (2a)^2 = 4a^2$, and now $dA/da = L$. Thus the derivative of the area of a square with respect to the apothem a is the perimeter, as Miller observed.

The change of variables in this example can be carried out in essentially any region! If $L = ks$ and $A = cs^2$, as above, set $a = 2A/L = (2c/k)s$. Then

$$L = k\left(\frac{ka}{2c}\right) = \frac{k^2}{2c}a \qquad \text{and} \qquad A = c\left(\frac{ka}{2c}\right)^2 = \frac{k^2}{4c}a^2,$$

and now $dA/da = L$. The linear dimension $a = 2A/L$ thus plays the role of the radius or apothem.

The derivative of the area is therefore the perimeter for any region, when the derivative is taken with respect to an appropriate linear dimension (and we have indicated how to determine that dimension). All we require is that the perimeter and area of the region both be well-defined and finite. For example, any *Jordan region* (a plane region bounded by a simple closed rectifiable curve) has finite area and perimeter, given by the usual integral formulas [**1**]. A similar argument applies to show that the derivative of the volume V of a solid is the surface area S, if taken with respect to an appropriate linear dimension $a = 3V/S$. The results for circles and regular polygons, or spheres and regular polyhedra, are thus seen to be just special cases of a very general phenomenon.

References

1. Bart Braden, The surveyor's area formula, *College Mathematics Journal* 17:4 (1986) 326–337.
2. Jay I. Miller, Differentiating area and volume, *Two-Year College Mathematics Journal* 9:1 (1978) 47–48. Reprinted by the MAA in *A Century of Calculus*, Part II, 1992, pp. 134–135.