A Case of True Interest

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In this article, we illustrate how elementary results in the mathematics of finance can be used to solve a real-world problem.

Suppose a borrower secures two mortgages when purchasing a house. This practice is gaining in popularity, and plays an important role in creative financing. [See Michael Rosser's "Second Mortgages: Coming of Age," Mortgage Banking 44:2 (Nov. 1983) 21–28.] In a typical situation, the term of the first mortgage might run up to 40 years. The second mortgage, on the other hand, is for a term usually not exceeding 15 years. The nominal (quoted) interest rates for the two loans are usually different, the interest rate for the second mortgage being generally higher than the first mortgage.

What is the effective (true) rate of interest paid for the (combined) loan?

Before we can solve this problem, we review some notions from the mathematics of finance that will be needed in the sequel. Recall that *the effective* (or true) rate of interest is the rate which when compounded annually yields the same accumulated amount as the nominal rate compounded m times a year. Since \$1 compounded m times a year at the nominal rate r grows to $(1+(r/m))^m$ dollars, and \$1 compounded once annually at rate ρ becomes $1+\rho$ dollars, the relationship between r and ρ is

$$1+\rho=\left(1+\frac{r}{m}\right)^m. \tag{1}$$

Next, we will need an expression giving the monthly repayment towards a loan. If a loan amount A is to be paid off in n years at interest rate r per annum convertible monthly, then the monthly payment is

$$k=\frac{Ar}{12\left[1-\left(1+\frac{r}{12}\right)^{-12n}\right]}. \tag{2}$$

Equation (2) is normally obtained by treating the present amortization problem as a special case of the annuity problem. Equivalently, we can derive (2) by observing that the amount of money due the finance company at the end of year n is

$$A\left(1+\frac{r}{12}\right)^{12n}-k\left[1+\left(1+\frac{r}{12}\right)+\cdots+\left(1+\frac{r}{12}\right)^{12n-1}\right].$$

Setting this expression equal to zero (we wish to amortize the loan by the end of year n) and using the formula for summing a geometric series yields the desired result.

We are now in a position to determine the effective interest rate for two different loans—as, for example: A_i = amount of loan i; n_i = term (in years) of loan i; r_i = nominal interest rate per annum, convertible monthly, for loan i; k_i = size of each monthly installment for loan i. From (2), we have

$$k_1=\frac{A_1r_1}{12\left[1-\left(1+\frac{r_1}{12}\right)^{-12n_1}\right]} \quad\text{and}\quad k_2=\frac{A_2r_2}{12\left[1-\left(1+\frac{r_2}{12}\right)^{-12n_2}\right]}. \tag{3}$$

Let R denote the nominal rate of interest for the (combined) loan convertible monthly. Assuming that $n_1>n_2$ and that the accumulated $12n_2$ payments, each of amount k_2, are reinvested during the remaining $12n_1-12n_2$ periods, we have

$$k_1\left[\left(1+\frac{R}{12}\right)^{12n_1-1}+\left(1+\frac{R}{12}\right)^{12n_1-2}+\cdots+1\right]+k_2\left[\left(1+\frac{R}{12}\right)^{12n_1-1}+\cdots+\left(1+\frac{R}{12}\right)^{12(n_1-n_2)}\right]$$
$$=(A_1+A_2)\left(1+\frac{R}{12}\right)^{12n_1}. \tag{4}$$

(The sum on the left-hand side of (4) gives the total amount the financier would have finished with if he had reinvested each repayment collected over the remaining term of the loan.) Using the formula for summing a geometric series, and rearranging, we obtain

$$R = f(R), \tag{5}$$

where

$$f(R) = \frac{12}{A_1 + A_2}\left\{k_1\left[1 - \left(1 + \frac{R}{12}\right)^{-12n_1}\right] + k_2\left[1 - \left(1 + \frac{R}{12}\right)^{-12n_2}\right]\right\}.$$

It is not difficult to show that there exists a unique solution to Equation (5). Indeed, for $R > 0$, it is easily verified that $f(R)$ is monotonically increasing and concave downward. Moreover, $f(R) < 12$. Therefore, $y = f(R)$ and $y = R$ intersect for some unique value $R \in (0, 12)$. Once the value of R has been found, the corresponding effective interest rate ρ may be found by taking $m = 12$ in (1). To solve (5), we iterate

$$R_{i+1} = f(R_i) \tag{6}$$

with a starting point (initial guess) $R_0 \in (r_1, r_2)$. Note that convergence of the iteration, independent of the choice of R_0 (as long as it is positive) is assured by the monotonicity and concavity of $y = f(R)$ in the interval $(0, \infty)$.

Example. Suppose you secure two mortgages when purchasing a house. The first mortgage is for a sum $A_1 = \$80{,}000$ to be amortized over $n_1 = 30$ years with equal monthly installments. The second mortgage is for a sum $A_2 = \$20{,}000$ to be repaid in equal monthly installments over $n_2 = 10$ years. The interest rate for the first loan is $r_1 = 12\%$ per annum convertible monthly, while the interest rate for the second loan is $r_2 = 14\%$ per annum convertible monthly. Determine the true rate of interest you are paying for the two mortgages.

Solution. Using (3), we determine

$$k_1 = \frac{(80{,}000)(0.12)}{12\left[1 - \left(1 + \frac{0.12}{12}\right)^{-360}\right]} = 822.89009 \quad \text{and} \quad k_2 = \frac{(20{,}000)(0.14)}{12\left[1 - \left(1 + \frac{0.14}{12}\right)^{-120}\right]} = 310.53311.$$

Next, using (5) and (6), we iterate

$$R_{i+1} = 0.00012\left\{822.89009\left[1 - \left(1 + \frac{R_i}{12}\right)^{-360}\right] + 310.53311\left[1 - \left(1 + \frac{R_i}{12}\right)^{-120}\right]\right\}$$

with say, $R_0 = 0.13$. With the aid of a pocket calculator, we obtain the following sequence:

$$\begin{array}{llll} R_1 = 0.1237427 & R_2 = 0.1226728 & R_3 = 0.1224774 & R_4 = 0.1224411 \\ R_5 = 0.1224347 & R_6 = 0.1224335 & R_7 = 0.1224333 & R_8 = 0.1224333. \end{array}$$

Thus, we may take $R = 0.1224$. From (1), we now compute the true rate of your \$100,000 loan over the 30 year term to be 12.9% per annum. (Note also that the true rates of the first and second mortgages over their respective terms are 12.68 and 14.93 per cent per annum, respectively.)

Remark. In this example, we have purposely omitted certain terms such as pre-paid finance charges and payment for private mortgage insurance premiums. These items may be easily incorporated into our model.