A Discrete Intermediate Value Theorem

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Many theories in mathematics (for instance, difference and differential equations) come in discrete and continuous versions. Indeed, an entire book, *Excursions in Calculus: An Interplay of the Continuous and the Discrete* by Robert M. Young (MAA, Washington, DC, 1992), has been devoted to this topic.

In this note, I give a discrete intermediate value theorem and apply it in an appealing induction proof.

The continuous intermediate value theorem is well known:

If f is continuous on $[a, b]$ and $f(a)f(b) < 0$, then $f(x) = 0$ for some x in (a, b).

A discrete intermediate value theorem (DIVT) is:

Let f be an integer-valued function defined on the integers in $[m, n]$. Suppose (as the equivalent of a continuity assumption) that $|f(i) - f(i+1)| \leq 1$ for $m \leq i < n$. If $f(m)f(n) < 0$, then $f(x) = 0$ for some integer x in (m, n).

As an application of the DIVT, consider the set L of all strings of the symbols a and b that can be constructed from the null string by application of a finite sequence of the rules:

1. if $\alpha \in L$ then $a\alpha b \in L$ and $b\alpha a \in L$;
2. if $\alpha \in L$ and $\beta \in L$ then $\alpha\beta \in L$.

It is clear from this definition that every element of L has equal numbers of a's and b's. But what about the converse?

Let α be a string of length n with equal numbers of a's and b's. We use induction on n to show that α is in L. The base case, $n = 0$, is the null string, which by definition is in L. For the inductive step, assume that $n > 0$ and that any string of length less than n with equal numbers of a's and b's is in L.

First suppose that α starts with a and ends with b. That is, $\alpha = a\beta b$ for some string β. Then β has equal numbers of a's and b's. So, by the inductive assumption, β is in L. It now follows that α is in L by rule 1. By the same argument, if α starts with b and ends with a, α is in L.

Now suppose that α starts and ends with a (and still has equal numbers of a's and b's). Here is where the DIVT can be used. Let

$$f(i) = \text{number of } a\text{'s} - \text{number of } b\text{'s}$$

within the first i symbols of α. Then $f(1) = 1$ and $f(n-1) = -1$.

Since $f(i+1)$ and $f(i)$ always differ by 1, f satisfies the "continuity" assumption, and the DIVT says that $f(i) = 0$ for some i between 1 and $n - 1$. It follows that $\alpha = \beta\gamma$, where β and γ are non-null strings having equal numbers of a's and b's. By the inductive assumption, β and γ are in L and, hence, $\beta\gamma$ is in L by rule 2. Similarly, if α starts and ends with b, α is in L.

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