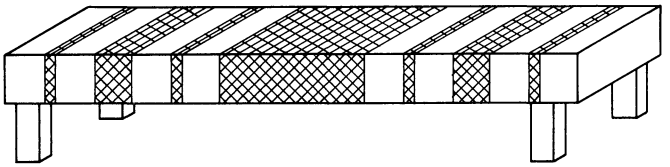

Amount of table removed: $\frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \frac{1}{32} + \cdots$

Remaining lengths less than: $\frac{l}{2}, \frac{l}{4}, \frac{l}{8}, \frac{l}{16}, \cdots$

Since the total amount of table taken away is the sum of the above geometric series, exactly one-half of the table has been removed, as promised. On the other hand, the length of any remaining piece is no less than 0 and no greater than the limit of the sequence above—which is 0. There are no intervals of table of nonzero length left; the table has disappeared.

A moment of silence is the standard response to these conclusions. Is there really any table left? Because of the construction it does not seem worthwhile to try to identify any of the remaining points. It is easy, however, to show that there is still table present. The only requirement is an object longer than $1/4$ the length of the table. (For short tables a calculus book is an excellent prop.) Simply put the object on the table and ask if it will fall. To fall, the object must go through a section that has been removed. Since the object is longer than $l/4$, it will not fit through any of the removed sections. No, the object will not fall; there is still table present and, in fact, $1/2$ of the table is still there. There are simply no intervals of nonzero length left.

As a follow-up, ask students to show how a table can be made to disappear by removing $1/4$ of the table. Can the same thing be done by removing $1/10$ of the table? This demonstration takes only 10 minutes of class time, plus a bit of outside work by students, if desired. Students will accept the conclusions—the logic is too simple and straightforward—but they will not be comfortable with them: half the table remains, but there are no intervals of nonzero length! Mysteries lie hidden within simple ideas and there may be more to mathematics than they have imagined. Demonstrating this to students is worth the time spent.

Bernoulli's Inequality and the Number e

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Our purpose is to give an elementary proof of the fact that $S_n = (1 + 1/n)^n$ increases and $T_n = (1 + 1/n)^{n+1}$ decreases to the same limit, without using the concept of the integral or properties of the natural logarithm.

The following remarks are essential:

(i) Since $T_n = (1 + 1/n)S_n$ and $(1 + 1/n) \to 1$ as $n \to \infty$, the sequences S_n and T_n cannot have different limits.

(ii) Since $T_n > 0$, the sequence T_n is bounded from below.

Therefore, it remains to prove that T_n is decreasing. Our argument is based on Bernoulli's inequality: for $x > -1$, $x \neq 0$ and all natural $n > 1$,

$$(1 + x)^n > 1 + nx. \qquad (*)$$

One proof of (*) is by induction. Clearly $(1+x)^2 > 1+2x$. And if $(1+x)^k > 1+kx$, then

$$(1+x)^{k+1} = (1+x)^k(1+x) > (1+kx)(1+x) = 1+(k+1)x+kx^2 > 1+(k+1)x.$$

To prove that $T_n = \left(1+\frac{1}{n}\right)^{n+1}$ is decreasing, consider

$$\frac{T_{n-1}}{T_n} = \left(\frac{n^2}{n^2-1}\right)^n \cdot \left(\frac{n}{n+1}\right) = \left(1+\frac{1}{n^2-1}\right)^n \cdot \left(\frac{n}{n+1}\right)$$

for $n > 1$. By virtue of (*),

$$\left(1+\frac{1}{n^2-1}\right)^n > 1+\frac{n}{n^2-1} > 1+\frac{n}{n^2} = \frac{n+1}{n},$$

we obtain

$$\frac{T_{n-1}}{T_n} > \left(\frac{n+1}{n}\right) \cdot \left(\frac{n}{n+1}\right) = 1.$$

Thus, $T_{n-1} > T_n$.

Since $S_n < T_n$ for all n, and $T_1 = 4$, it follows that S_n is bounded from above. To show that S_n is increasing, observe that

$$\frac{S_n}{S_{n-1}} = \frac{\left(\frac{n+1}{n}\right)^n \cdot \left(\frac{n}{n-1}\right)}{\left(\frac{n}{n-1}\right)^n} = \left(\frac{n^2-1}{n^2}\right)^n \cdot \left(\frac{n}{n-1}\right) = \left(1-\frac{1}{n^2}\right)^n \cdot \left(\frac{n}{n-1}\right)$$

for $n > 1$. Again, using (*), we get

$$\left(1-\frac{1}{n^2}\right)^n \cdot \left(\frac{n}{n-1}\right) > \left(1-\frac{n}{n^2}\right) \cdot \left(\frac{n}{n-1}\right) = 1.$$

Hence, $S_n > S_{n-1}$.

Editor's Note: For a related discussion using the Fundamental Theorem of Calculus and the definition of e as the unique number for which $\int_1^e dx/x = 1$, see Lee Badger's classroom capsule "A Nonlogarithmic Proof that $(1+1/n)^n \to e$" [TYCMJ, 13(November 1982) 331–332].

Area of a Parabolic Region

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Although students can quickly recall that the area of a circle is πr^2 and the area of an ellipse is πab, there does not appear to be a standard formula that they can recall when dealing with areas of parabolic regions. Thus it may be instructive to prove the following: