Parametric Integration Techniques

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In this note, we present an integration technique that evaluates integrals through the manipulation of a parameter. By "parametric integrals" we mean single integrals of bivariate functions with respect to one of the variables (the variable of integration), while the other variable is referred to as a *parameter*. By "manipulation" we mean operations under the integral sign in terms of the parameter, such as: differentiation, integration, or some other kind of limiting processes (like summation over an infinite index set, etc.). We illustrate the method with the help of some selected examples. These examples will show that the parametric integration technique only requires the mathematical maturity of Calculus III level and provides a very straightforward method to evaluate difficult integrals for which one commonly uses the method of *contour integration*. However, this technique does not appear to be widely used. We begin with a simple example.

Example 1. Consider the well-known integral

$$\int_0^{\infty}\frac{dx}{p^2+x^2}=\frac{\pi}{2p},\qquad (p>0).$$

Considering p as a parameter and differentiating with respect to it, we obtain

$$\int_0^{\infty}\frac{-2p\,dx}{\left(p^2+x^2\right)^2}=-\frac{\pi}{2p^2},\qquad (p>0)$$

or

$$\int_0^{\infty}\frac{dx}{\left(p^2+x^2\right)^2}=\frac{\pi}{4p^3},\qquad (p>0)$$

as a new integral formula. The standard method to derive this is integration by parts, but that would take a bit longer.

Continuing this process of differentiating with respect to p and simplifying afterwards, one obtains the formulas for the values of the integrals of $1/(p^2+x^2)^3$, $1/(p^2+x^2)^4$, etc., or $1/(p^2+x^2)^n$ by induction.

A natural question arises: How should one introduce a parameter within the integrand? Usually, this is not a problem. In many integrals, especially in the formulas of integral transforms, parameters are already present. However, there exist cases when at the outset an integral may contain no parameter. In such cases, a parameter is generally introduced by changing the variable of integration using a substitution that contains a parameter. For example, consider the well-known integral

$$\int_0^{\infty}dx/(1+x^2)=\pi/2$$

(a special case of the integral in Example 1 above with $p=1$). In this example, we can introduce a parameter p by substituting x/p for x.

The application of limiting processes under the integral sign requires the interchangeability of two limiting processes, one being the integration itself and the other one a limiting process (differentiation or integration, etc.) in terms of the parameter. More precisely, we must assume the following:

(1) Differentiation with respect to a parameter under the integral sign is permitted.
(2) Integration with respect to a parameter under the integral sign is permitted.
(3) Limit and an integral sign can be interchanged.

Theorems establishing conditions when the order of these processes may be interchanged have long been known; some can be traced back to Leibniz. Studying these (see [3], pp. 286–292), we shall find that the conditions are quite general (that is, general enough to cover the evaluation of a broad class of integrals). The *uniformity* of the limiting processes is a usual requirement, but this is evident in most cases when the integrals themselves exist. However, care must be exercised to avoid absurdities (see Example 6 below).

We present the rest of the examples under the above assumptions.

Example 2. Show that $\Gamma(n+1)=\int_0^\infty x^n e^{-x}\,dx=n!$ for a nonnegative integer n.

We begin with the well-known integral

$$\int_0^\infty e^{-ax}\,dx=\frac{1}{a},\qquad (a>0).$$

Repeated differentiation with respect to the parameter 'a' gives

$$\int_0^\infty xe^{-ax}\,dx=\frac{1}{a^2},$$

$$\int_0^\infty x^2e^{-ax}\,dx=\frac{2}{a^3},$$

$$\vdots$$

$$\int_0^\infty x^ne^{-ax}\,dx=\frac{n!}{a^{n+1}}.$$

Choosing $a=1$, we obtain the desired formula.

Example 3. Show that

$$\int_0^1 x^n(\log x)^k\,dx=\frac{(-1)^k k!}{(n+1)^{k+1}},\qquad (n\neq -1).$$

We begin with the well-known formula

$$\int_0^1 x^n\,dx=\frac{1}{n+1},\qquad (n\neq -1).$$

Considering n as a parameter instead of a constant and differentiating both sides with respect to it, we obtain

$$\int_0^1 x^n\log x\,dx=-\frac{1}{(n+1)^2}=\frac{(-1)^1\cdot 1!}{(n+1)^{1+1}}.$$

Thus, the result holds for $k=1$. To obtain the formula for the general case, we continue differentiating and apply induction.

Example 4. Show that

$$\int_0^\infty \frac{\sin x}{x}\,dx = \frac{\pi}{2}.$$

This integral is generally evaluated by using contour integration and thus requires the theory of complex functions. However, it can be computed easily by applying parametric integration. To show this, we need to consider the integral of the product of $\sin(x)$ with another function involving a parameter that will:

(i) guarantee the existence (convergence) of the integral over the interval $(0, \infty)$ and

(ii) render a denominator of x from integrating under the integral sign with respect to the parameter p.

An integral satisfying these requirements is readily available in the form of the Laplace transform of $\sin x$, that is

$$\int_0^\infty e^{-px}\sin x\,dx = \frac{1}{1+p^2}, \qquad (p > 0). \qquad (*)$$

Integrating both sides with respect to the parameter from 0 to p, we obtain

$$\int_0^\infty \frac{1-e^{-px}}{x}\sin x\,dx = \arctan p.$$

The last formula is almost what we wanted, with the exponential term yet to be removed. To this end, we apply the limit as $p \to \infty$.

Example 5. We can use Example 4 to show that

$$\int_0^\infty \left(\frac{\sin t}{t}\right)^2 dt = \frac{\pi}{2}.$$

Substituting $t = \frac{x}{p}$ $(p > 0)$ in Example 4, we obtain

$$\int_0^\infty \left(\frac{\sin pt}{t}\right) dt = \frac{\pi}{2}, \qquad (p > 0). \qquad (**)$$

Integrating both sides with respect to the parameter from 0 to p, we obtain

$$\int_0^\infty \left(\frac{1-\cos pt}{t^2}\right) dt = \frac{\pi p}{2}. \qquad (***)$$

Finally, substituting $p = 2$ and dividing both sides by 2, we obtain the result,

$$\int_0^\infty \left(\frac{\sin t}{t}\right)^2 dt = \frac{\pi}{2}.$$

Example 6. In this example, we illustrate why care must be exercised to avoid absurdities. Consider the integral given by $(**)$ in Example 5 above, that is

$$\int_0^\infty \frac{\sin pt}{t}\,dt = \frac{\pi}{2}, \qquad (p > 0).$$

Differentiating both sides with respect to the parameter p, we obtain

$$\int_0^\infty \cos pt\, dt = 0,$$

which is absurd as the integral on the left-hand side does not converge.

The true convenience of parametric integration becomes most evident when one compares the use of parametric integration with the derivations resulting from contour integration. Among others, the list of integrals we have checked includes the following:

$$\int_0^\infty \left(\frac{x^{-\alpha}}{1+x} \right) dx = \frac{\pi}{\sin \pi\alpha}, \qquad (0 < \alpha < 1) \tag{1}$$

$$F_1 = \int_{-\infty}^\infty \sin x^2\, dx = \sqrt{\frac{\pi}{2}}, \tag{2}$$

$$F_2 = \int_{-\infty}^\infty \cos x^2\, dx = \sqrt{\frac{\pi}{2}}, \tag{3}$$

$$\int_0^\infty \frac{\cos px}{1+x^2}\, dx = \frac{\pi}{2} \cdot e^{-p}, \qquad (p \geqslant 0) \tag{4}$$

$$F(p) = \int_0^\infty \frac{\sin px}{x(1+x^2)}\, dx = \frac{\pi}{2} \cdot (1 - e^{-p}), \qquad (p \geqslant 0). \tag{5}$$

Formula (1) is extremely useful. It embodies a host of other integral formulas by assigning specific numerical values to α and changing the integration variable. To derive it, we consider the geometric series expansions of the function $f(x) = 1/(1+x)$, that is

$$1/(1+x) = 1 - x + x^2 - x^3 + \cdots,$$

and

$$1/(1+x) = \frac{1}{x} - \frac{1}{x^2} + \frac{1}{x^3} - \frac{1}{x^4} + \cdots,$$

according to whether $|x|$ is less than 1 or greater than 1. Hence, we split the interval of integration in two parts, namely $(0,1)$ and $(1,\infty)$, and evaluate the integrals of the infinite series termwise. In this example, the role of parameter is taken by the discrete variable summation index. The result is the partial fraction series of $\pi/\sin \pi\alpha$.

The integrals F_1 and F_2 are known as Fresnel integrals. We sketch the derivation of F_1; the derivation of F_2 is analogous. Substituting $t = x^2$ in F_1, we obtain

$$F_1 = \int_0^\infty \frac{\sin t}{\sqrt{t}}\, dt.$$

Replacing p by p^2 in formula $(*)$ above, we obtain

$$\int_0^\infty e^{-p^2 t} \sin t\, dt = \frac{1}{1+p^4}.$$

Integrating both sides with respect to p from 0 to infinity gives

$$\int_0^\infty \sin t \left(\int_0^\infty e^{-p^2 t}\, dp \right) dt = \int_0^\infty \frac{1}{1+p^4}\, dp.$$

Since

$$\int_0^\infty e^{-p^2 t}\,dp = \frac{1}{2}\sqrt{\frac{\pi}{t}}$$

(a well-known formula from probability theory) and

$$\int_0^\infty \frac{dp}{1+p^4} = \frac{\pi}{2\sqrt{2}},$$

(obtained from (1) above by substituting p^4 for x and $3/4$ for α), the value of F_1 is evident.

Formulas (4) and (5) are interrelated; one can be obtained from the other by integration or differentiation, respectively, with respect to the parameter p. Differentiating the integral (5) with respect to p twice, we can easily verify that it satisfies the differential equation:

$$F''(p) - F(p) = -\frac{\pi}{2},$$

with initial values $F(0) = 0$ and $F'(0) = \pi/2$. The unique solution of this differential equation provides the value of the integral (5).

These and other examples clearly demonstrate the power and efficiency of the parametric technique as well as its superiority over alternative methods (e.g., contour integration). As a method, it is not unknown in the literature (see [1]–[7]) and every now and then there are some instances of its applications as cited in ([8]–[10]). However, it appears that the topic is generally not included in the undergraduate mathematics curriculum.

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