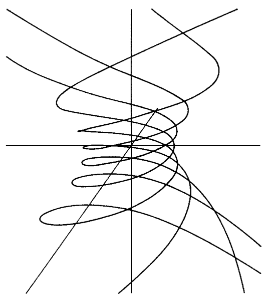

FIGURE 3.1
A projective helix lying on a hyperboloid of one sheet.

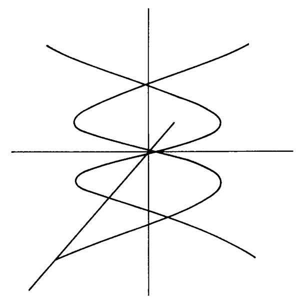

FIGURE 3.2
A projective helix with $p = 2$ and $q = \frac{1}{2}$.

group action is well defined on the space. Another example of such a space is relativistic space-time where the group acting is the Lorentz group. For a discussion of curves in this space, see [3].

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A Geometrical Approach to Cramer's Rule

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Gabriel Cramer (1704–52) published in 1750 [1] his celebrated Rule (already known to Maclaurin [2]) for solving simultaneous equations by means of determinants, which he had obtained by "the science of algebra." It may be of interest to consider how, upon certain assumptions, it may be obtained geometrically.

Let us start with the two simultaneous equations

$$ax + by = k$$
$$cx + dy = l$$

and write these in matrix form

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} k \\ l \end{bmatrix}.$$

Now consider the matrix as representing, not the usual *point* transformation, by which the point at the position (x, y) moves to, or maps into, the position (k, l), all coordinates referring to the same fixed axes, but a *coordinate* or *axis* transformation, by which the fixed point (k, l) is relabelled (x, y) in consequence of a change in the axes and so in the whole coordinate system. (The two transformations are mutually inverse.) The equation now states that the point with coordinates (k, l) relative to the original axes has coordinates (x, y) relative to the transformed axes with unit (basis) vectors (a, c) and (b, d).

(Note: As the basis vectors of a coordinate or axis transformation have unit length in that transformation, they are here referred to as "the unit vectors" of the transformation. Similarly the area or volume spanned by the unit vectors is referred to as "the unit area" or "the unit volume" of the transformation. The word "transformation" refers both to the process of obtaining one set of coordinates from another, and to the coordinate system resulting from that process.)

Now call the original axes $K'OK$ and $L'OL$, and call the points (k, l), (a, c) and (b, d) P, Q and R, respectively. Join OP, OQ and OR and complete the parallelograms $OQSR$ and $OPTR$. Extend TP to meet RS in M and OQ in N.

We now have P as the point with coordinates x and y relative to the transformed axes along OQ and OR, and since NP is parallel to OR,

$$x = \frac{\text{length of } ON}{\text{length of } OQ}$$

$$= \frac{\text{area of parallelogram ONMR}}{\text{area of parallelogram OQSR}}$$

(because the areas of parallelograms of the same height are proportional to the lengths of their bases)

$$= \frac{\text{area of parallelogram } OPTR}{\text{area of parallelogram } OQSR}$$

(because $ONMR$ and $OPTR$ have the same height and the same base OR)

$$= \frac{\begin{vmatrix} k & b \\ l & d \end{vmatrix}}{\begin{vmatrix} a & b \\ c & d \end{vmatrix}}$$

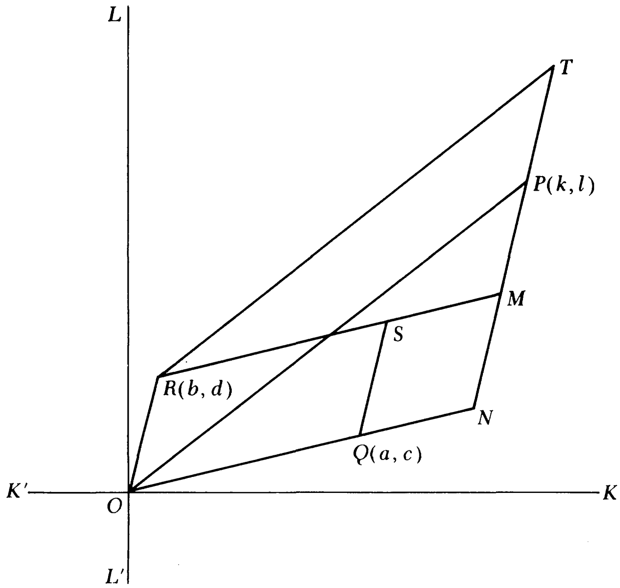
by the interpretation of a determinant as the area of the parallelogram spanned by its column vectors (the unit area of the corresponding transformation).

Similarly by completing the parallelogram on OP and OQ we find that

$$y = \frac{\begin{vmatrix} a & k \\ c & l \end{vmatrix}}{\begin{vmatrix} a & b \\ c & d \end{vmatrix}}.$$

We thus obtain Cramer's Rule for two equations in two unknowns.

The argument holds for all positions of P, Q, and R, provided account is taken of the sense of lines and the sense of description of areas. If OQ, OP and OR are simply rotated from the position in the diagram without transposition, the reasoning is unaffected. If two of them are transposed there is a reversal of (a) the sense of one or other of them, (b) the sense of description of the area spanned by them, and (c) the sign of that area. There are two special cases. If the columns of coefficients are dependent, so that O, Q and R are collinear, but (k, l) is independent, TP and OQ fail to meet, and there is no solution (the determinant of coefficients is zero). If (k, l)



is also dependent, so that P is also collinear, then TP and OQ coincide, and the position of N (determining the value of x) is arbitrary (both determinants are zero). The position of N also determines the value of y, which is given by the quotient of the lengths of NP and OR.

The extension of the argument to three dimensions seems fairly clear, given a suitable convention for determining sign. If we take the volume spanned by the original unit vectors to be positive, then the unit volume of the transformation could be considered positive or negative according to whether the number of pairs of unit vectors transposed by the transformation was even or odd. An equivalent criterion would be the number of unit vectors whose sense was reversed by the transformation. The special cases carry over with the substitution of "coplanar" for "collinear."

(Note: The result for two or three dimensions may be obtained very neatly by writing the original equations in column vector form, representing a coordinate or axis transformation, and taking vector products or scalar triple products. The procedure, however, is possibly more algebraic than geometrical.)

The further extension of the argument to higher dimensions is perhaps allowable by analogy. In each case the coordinate required is given by the quotient of two determinants, that in the denominator being the determinant of coefficients, and that in the numerator being the same except that the constant terms replace the coefficients of the required coordinate. In these determinants

(1) the columns of coefficients of coordinates other than the one required represent the vectors spanning the common base;

(2) the column of coefficients of the required coordinate represents the unit vector along the axis of that coordinate; and

(3) the column of constant terms represents the vector whose projection parallel to the base on the axis of the required coordinate gives the value of that coordinate.

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