A Surprise from Geometry

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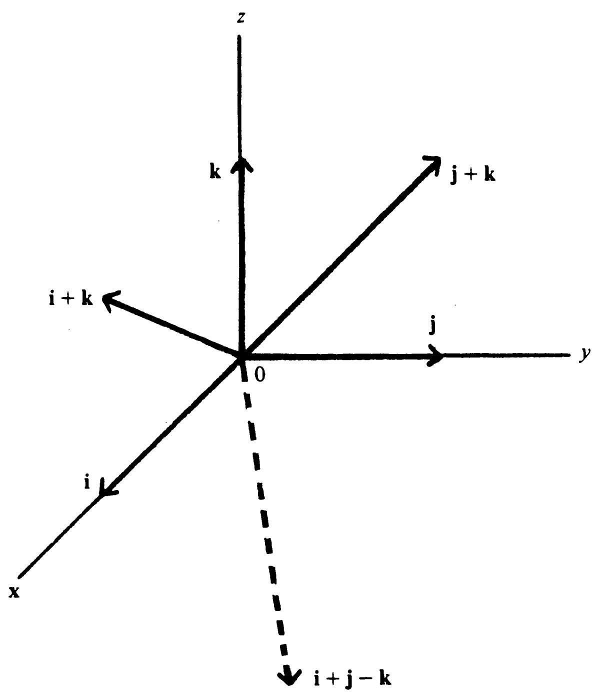
It is patently obvious that two vectors in the plane (all vectors are considered to issue from the origin), which meet at an angle that does not exceed a right angle, can be spun around the origin so that both vectors lie in the nonnegative quadrant (that is, the endpoint (x, y) of each vector has coordinates which are both nonnegative). It is not quite so obvious that a set of 3 vectors in 3-space, which in pairs meet at angles not exceeding a right angle, can always be spun around the origin to lie in the nonnegative octant. It is not at all obvious, but is also true, that any set of 4 vectors in 4-space, no 2 of which meet at an angle greater than a right angle, can be arranged to lie in the nonnegative orthant (orthant is the general term for quadrant and octant).

At this point, who can resist the conjecture that any set of n vectors in n-space, no 2 of which meet at an angle exceeding a right angle, can be rotated to a position so that all the vectors in the set are contained in the nonnegative orthant? Isn't it surprising that this is false for every $n > 4$?

We shall see that the set S, consisting of the five 3-dimensional vectors $\mathbf{i} = (1,0,0)$, $\mathbf{j} = (0,1,0)$, $\mathbf{i} + \mathbf{k} = (1,0,1)$, $\mathbf{j} + \mathbf{k} = (0,1,1)$, and $\mathbf{i} + \mathbf{j} - \mathbf{k} = (1,1,-1)$, cannot all lie in the nonnegative orthant of a space of any dimension.



First of all, it is easy to check that each pair of vectors in S meets at an angle not exceeding a right angle, and a glance at the figure shows that they fan out too far to fit into an octant of 3-space. We shall establish our general claim by arguing to a contradiction. Suppose, then, that in some n-space, our set S can be accommodated completely within the nonnegative orthant. Then the coordinates of each vector in S are all nonnegative. Since S does not contain the zero vector, none of these n-tuples of coordinates will consist entirely of 0's; in each case, at least one coordinate must actually be a positive number.

Now, the crux of our argument consists in showing that the positioning of S in the nonnegative orthant necessarily also brings into this orthant the companion vector $\mathbf{k} = (0, 0, 1)$, even though it does not belong to S.

While each coordinate of a vector in S is either positive or zero, a coordinate in the description of **k**'s position may presumably be positive, zero, or negative. Let us investigate the feasibility of a negative coordinate in **k**. If **k** were to have a negative coordinate in a component in which the vector **i** has a zero, then that component in their sum $\mathbf{i} + \mathbf{k}$ would have a negative value. But, since $\mathbf{i} + \mathbf{k}$ belongs to S, no component of $\mathbf{i} + \mathbf{k}$ is negative. Consequently, **k** can have a negative coordinate only in a position in which **i** has a positive coordinate (recall that the coordinates of **i** are either positive or zero). Similarly for the vector **j**: a negative component in **k**, opposite a zero in **j**, would yield a contradictory negative component in the vector $\mathbf{j} + \mathbf{k}$ of S. As a result, **k** can have a negative coordinate only in a place in which both **i** and **j** have a positive coordinate.

But there are no such places! If there were, such a pair of positive coordinates would contribute a positive amount t to the dot product $\mathbf{i} \cdot \mathbf{j}$. However, since **i** and **j** are orthogonal, we have $\mathbf{i} \cdot \mathbf{j} = 0$ in every coordinate system, yet there would be no way to nullify the above contribution t because there are no negative coordinates in any vector of S (in particular, in **i** and **j**). It follows, then, that **k** possesses no negative coordinates and must also reside in the nonnegative orthant.

Now we can conclude easily. Since both the vectors **k** and $\mathbf{i} + \mathbf{j} - \mathbf{k}$ have no negative coordinates, their dot product $\mathbf{k} \cdot (\mathbf{i} + \mathbf{j} - \mathbf{k})$ cannot be a negative number. However, obvious orthogonalities yield

$$\begin{aligned} \mathbf{k} \cdot (\mathbf{i} + \mathbf{j} - \mathbf{k}) &= \mathbf{k} \cdot \mathbf{i} + \mathbf{k} \cdot \mathbf{j} - \mathbf{k} \cdot \mathbf{k} \\ &= \ 0 \ + \ 0 \ - \ |\mathbf{k}|^2, \end{aligned}$$

which is negative, since **k** is not the zero vector, and the argument is complete.

This argument is due to the Israeli mathematician Moshe Roitman of the University of Haifa; it was most kindly communicated to me by his colleague Joe Zaks during his recent visit to Waterloo (summer, 1984). He also noted that L. M. Kelly and Shreedharan of Michigan State University in East Lansing had a similar example of a set of 5 vectors which lent itself to an easy argument based on inner products.

Proofs of the cases of vectors in 3-space and 4-space can be found in [**1**]. It is interesting that the concluding remark in this paper is an example of 8 vectors that need a space of at least 9 dimensions for their accommodation in the nonnegative orthant.

References

[1] L. J. Gray and D. G. Wilson, Nonnegative factorization of positive semidefinite nonnegative matrices, Linear Algebra and its Applications, 31 (1980) 119–127.