$$r_1(r_2+r_3):r_2(r_1+r_3):r_3(r_1+r_2)=a^2:b^2:c^2. \tag{17}$$

But this is exactly the system (6) that we encountered in the geometric problem. To apply the analysis we applied to (6), we first verify that certain quantities are nonzero. Let $\sigma_1, \sigma_2, \sigma_3$ be defined as in (3). From (17) we have, for some $\lambda > 0$,

$$r_1(r_2+r_3)=\lambda a^2, \qquad r_2(r_1+r_3)=\lambda b^2, \qquad r_3(r_1+r_2)=\lambda c^2. \tag{18}$$

Subtracting each equation from the sum of the other two gives

$$2r_2r_3=\lambda\sigma_1, \qquad 2r_1r_3=\lambda\sigma_2, \qquad 2r_1r_2=\lambda\sigma_3. \tag{19}$$

This shows that all the σ_i are positive, so the same analysis that led us from (6) to (8) can be applied to give, with $\tau_i = 1/\sigma_i$, $i=1,2,3$,

$$r_1:r_2:r_3=\tau_1:\tau_2:\tau_3. \tag{20}$$

Thus $r_i=\mu\tau_i$, $i=1,2,3$, for some $\mu>0$. The multiplier μ is determined by substituting into (16). We then obtain, with $T=\tau_1+\tau_2+\tau_3$,

$$r_1=\frac{a^2T}{\tau_2+\tau_3}, \qquad r_2=\frac{b^2T}{\tau_1+\tau_3}, \qquad r_3=\frac{c^2T}{\tau_1+\tau_2}. \tag{21}$$

This shows that the r_i are uniquely determined by a^2, b^2, c^2, as claimed.

Gilbert and Shepp [**2**] showed that in the generalization to n resistances $r_1,\dots,r_n$ on the sides of an n-gon, the r_i are uniquely determined by the equivalent resistances between consecutive vertices.

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Some Remarks About Bridge Probabilities

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This paper is dedicated to the memory of Mark Kac, great mathematician and probabilist.

Probability theory is basic in many areas of physics and engineering, for example, quantum theory, statistical mechanics, and risk-benefit analysis. I have found it valuable in my teaching to go back to some very basic concepts and to discuss them in detail with the students, hoping that the proper building blocks will be available for constructing the more complicated situations necessary to describe, for instance, physical systems in statistical mechanics. Since my avocation has been, since my student days long ago, various games of chance (principally, backgammon and bridge), I find it useful to choose my examples from these areas. This also has the advantage of retaining the students' interest longer than, say, a laborious discussion of molecules confined in a cell of phase space. I felt it might be worthwhile to share some of these examples which, by and large, are not found in the standard elementary texts, with the readers of this journal.

A sample problem which arises in backgammon is: *given an arbitrary roll of two unbiased dice, what is the probability that a given number, say ⚀, will appear on one or both of the dice?* The correct answer of 11/36 can be arrived at by brute force (i.e., enumerate all possible configurations) or by the formula

$$P(A \vee B) = P(A) + P(B) - P(A \wedge B). \tag{1}$$

But the practical solution is simply to calculate the probability that a 1 does *not* appear. Almost anybody will come up with $(5/6)^2$ for that one, whereupon subtraction from 1 gives the correct answer.

In this trivial example, direct enumeration or application of the abstract formula may seem to be just about as easy as the *practical* solution, but let's complicate the problem a little by asking about n rolls. Then it is very hard to arrive at the result $1-(5/6)^n$ in any other way, *especially for arbitrary* n.

This method of calculation is actually based on De Morgan's Law $(A \vee B) = (A' \wedge B')$; since $P(A') = 1 - P(A)$ it follows that

$$P(A \vee B) = 1 - P(A')P(B') \tag{2}$$

because $P(A \wedge B) = P(A)P(B)$ always holds for independent events A and B. In fact, (2) holds *only* if A and B are independent. If A and B are *not* independent, $P(A \vee B) = 1 - P(A')P(B'|A')$ where $P(B'|A')$ is the conditional probability of B' given A'.

In my favorite pastime, bridge, a player is frequently called upon to choose among several lines of play; of course, the winning player will choose the line with the greatest probability of success. Suppose a given line will succeed if spades break 3–3 (each of the two *opponents* holds three cards of the spade suit) with probability p_1 or if West holds the king of hearts with probability p_2 or if the jack of spades lies in the hand with no trumps with probability p_3. Then the probability of success is approximately calculated as $1-(1-p_1)(1-p_2)(1-p_3)$; the calculation is approximate because the events may not be strictly independent.

I know that all bridge players (at least those who bother to calculate at all) have learned to calculate this way, because there is really no other way to do it. Now consider a slightly different situation: the line of play will succeed if spades break 4–2 (probability p_1) or 3–3 (probability p_2). These two events are evidently mutually exclusive so referring to (1) we see that the probability is $p_1 + p_2$. One can then go on to pose problems in which one must decide whether given events are independent or mutually exclusive or somewhere in between, and compute (or estimate) the probabilities in each case. All three situations arise in bridge; as a practical matter, only the two extremes are usually considered in actual play, during which only mental calculations are allowed by the rules of the game.

I would next like to illustrate the point that very often conditional probability is a more useful tool for carrying out practical computations than is absolute probability. In the course of this discussion, I shall demonstrate an effective and simple method for computing the probabilities of various bridge-suit breaks.

Suppose you have two red balls and two blue balls and place them at random, two each, in two boxes. What is the probability that both blue balls will appear in the same box? In this simple case, a direct enumeration is possible. Using an obvious notation, we have B_1B_2, B_1R_1, B_1R_2, B_2R_1, B_2R_2 and R_1R_2 as the equally likely possible contents of, say box one, from which we see that the desired probability is 1/3. This is, of course, the absolute approach. The conditional approach is to assume that one of the blue balls, say B_1, is placed in one of the boxes, e.g., box number one. The remaining three balls are dealt at random. Clearly ball B_2 has twice as much chance of going into box number two, with two vacancies, as into box one which has only one vacancy remaining, so we again arrive at the correct answer 1/3. If the conditional computation seems to have little advantage over the absolute in this case, modify the problem just a bit to ask a bridge question. If thirteen cards are dealt to each of two players, East and West, and if among the twenty-six cards dealt are two spades, what is the probability that both spades are in the same hand, i.e., that spades break 2–0? Absolute computation now becomes lengthy while the conditional calculation is just as fast as in the two-ball case, easily giving the answer $12/25 = 48\%$.

Similarly, for three outstanding cards the probability of a 3–0 break is $(12/25) \cdot (11/24) = 22\%$ while a 2–1 break occurs in $3 \cdot (13/25) \cdot (12/24) = 78\%$ of the cases (the factor 3 here corresponds to the number of ways that a 2–1 break can occur, 6, times 1/2, since the first card can be placed in either hand). Following similar reasoning, we can calculate the probability of 4–0, 3–1 and 2–2 breaks with four cards outstanding as $11/115 = 9.6\%$; $286/575 = 49.7\%$; and $234/575 = 40.7\%$, respectively. A relatively simple rule for calculating the probability of an m–n break is the following. Calculate the number of ways the break can occur, multiply by 1/2, since either hand can hold the first card, and proceed to multiply by

$$\frac{12}{25} \cdot \frac{11}{24} \cdots \frac{13-(m-1)}{26-(m-1)} \cdot \frac{13}{26-m} \cdots \frac{13-(n-1)}{26-(m+n-1)}.$$

These are just the appropriate factors corresponding to "filling up" vacancies in the two hands. It is easy to verify that it makes no difference in which order one assumes the vacancies are filled. (Incidentally, the number of ways the m–n break can occur is given by $2C(m+n, n)$ if $n \neq m$ but by $C(m+n, m)$ if $m = n$, where $C(k, j)$ is the number of combinations of k objects taken j at a time.

It appears that this approach to computing suit division is not well known, even to the experts. Thus Hugh Kelsey, in chapter 6 of the otherwise outstanding book *Advanced Play at Bridge* [**5**] suggests that the probability of a 4–2 break be computed combinatorically as

$$\frac{2 \cdot C(6,2) \cdot C(20,11)}{C(26,13)} = 2\left[\frac{6!}{4!2!} \cdot \frac{20!}{11!9!}\right] \Big/ \frac{26!}{13!13!} = \frac{2519400}{10400600}.$$

This, of course gives the same answer (48.4%) as the formula I have suggested,

$$30 \cdot \frac{1}{2} \cdot \frac{12}{25} \cdot \frac{13}{24} \cdot \frac{12}{23} \cdot \frac{11}{22} \cdot \frac{10}{21},$$

except that Kelsey's approach seems to require somewhat more effort. Incidentally, a "quick and dirty" answer can be obtained by dividing the number of ways six cards can be distributed between two hands 4–2 ($2C(6,2) = 30$) by the total number of ways six cards can be distributed ($2^6 = 64$) to obtain 46.8% or, for "at table" mental calculations, $15/32 \approx 1/2$.

Probably the least intuitively understood aspect of probability theory involves applications of Bayes' Theorem. Recall that Bayes' Theorem tells us that if a probability measure space S is divided into n disjoint sets A_i, and if $B \subset S$ then

$$P(A_l|B) = \frac{P(B|A_l)P(A_l)}{\sum_i P(B|A_i)P(A_i)}, \qquad l = 1, 2, \dots n, \tag{3}$$

where $P(A|B)$ represents the conditional probability of A given B. This relation is easy to derive from the two basic formulas

$$P(B) = \sum_{i=1}^{n} P(A_i)P(B|A_i)$$

and $P(A \wedge B) = P(B|A)P(A) = P(A|B)P(B)$.

A well-known nonbridge example of Bayes' Theorem is the so-called jailer's paradox. In this scenario, A, B, and C are in prison; one of the three is condemned to death while the other two will be freed, but only the jailer knows which. Now A wants to send a letter to his girl friend, and since he knows that with certainty either B or C is going free (remember, only one of the three will die) he asks the jailer to tell him the name of one of the other two who will be freed and thus would be able to deliver the letter (should A turn out to be the victim). The jailer demurs, explaining that as of now A has $1/3$ probability of dying but if he were to say, for example, that B would go free, then A's probability of dying would increase to $1/2$.

This seems intuitively suspect, since the jailer is providing no new information to A, so why should his probability of dying change? Here, Bayes' Theorem gives the correct answer (still $1/3$) easily: set A_1 = "A dies," A_2 = "C dies," and B = "Jailer says B goes free." Note: $P(B|A_1) = 1/2$ but $P(B|A_2) = 1$.

Aside from the formal application of Bayes' Theorem, one would like to understand this "paradox" from an intuitive point of view. The crucial point which can perhaps make the situation clear is the discrepancy between $P(B|A_1)$ and $P(B|A_2)$ noted above. Bridge players call this the "Principle of Restricted Choice" (more on this later). The probability of a restricted choice is obviously greater than that of a free choice, and a common error made by those who attempt to solve such problems intuitively is to overlook this point. In the case of the jailer's paradox, if the jailer says "B will go free" this is twice as likely to occur when C is scheduled to die (restricted choice; jailer *must* say "B) as when A is scheduled to die (free choice; jailer could say either "C" or "B").

The history of this principle in the game of bridge is extremely interesting. The prototype application occurs in the following situation. South, declarer, holding the ace, king and one or more other spades, plays the ace of spades. There are four spades outstanding between the defenders, East and West, the "2", "3", queen(Q) and jack(J). East plays the 2 and West the Q. South must now decide who has the J, so that he can play appropriately (finesse, or drop) in order to avoid losing a trick to the jack of spades. (The reader need know nothing about bridge in order to understand the probabilistic situation. The only point needed is that the queen and jack are equals; holding both, a (good) defender would play each one with probability $1/2$. Also, holding a low card $(3,2)$ in addition to a high card or cards (Q, J) a defender would invariably play the low card. Of course, a player is required to "follow suit" if able, i.e., when South plays a spade, the other players must play spades if they hold any.)

Let's look at a diagram of the situation. The possible holdings after the play of one round of spades are

	West	*East*
1)	J	3
2)	None	$J, 3$

Since there are only two cases, it seems obvious that the probability that West also holds the jack is $1/2$. Even very well trained mathematicians often cannot bring themselves to believe that the probability of case 1) above is not $1/2$, but $1/3$, without going through the detailed calculation. (To be precise, we have to take into account the probability of a 2–2 break vs. a 3–1 break as computed earlier. I will leave it as an exercise in the application of Bayes' Theorem to show that the precise probability of West's holding the Q, J is $6/17$ rather than $1/3$.) The point is, of course, that scenarios 1) and 2) above are *not* equally probable. Scenarios 1) and 2) are induced by the original holdings

	West	*East*
1′)	Q, J	$3, 2$
2′)	Q	$J, 3, 2$

In Case 1′, West would have played the jack $1/2$ of the time, whereas in Case 2′ his choice is *restricted* to playing the queen. Thus Case 2 is twice as probable as Case 1. From the point of view of Bayes' Theorem, $P(\text{West plays } Q|\text{West holds } Q, J) = 1/2$, whereas $P(\text{West plays } Q|\text{West holds only } Q) = 1$.

This is a subtle and easily overlooked point. Contract bridge has been played since 1929 and its predecessors (auction bridge, dummy bridge, and whist) several hundred years more. The same problem arises in all of these games, and yet, until approximately 1960, the two cases were treated as equally probable, even by expert players. Since expert bridge players always go with the odds (as they perceive them) this indicates how difficult the odds may be to figure when Bayes' Theorem is involved.

It is interesting that the correct odds for this, and similar bridge situations, were not calculated originally from Bayes' Theorem, but were estimated from the following ingenious heuristic argument. Since it is normal, in bridge, when not attempting to win a trick, to follow suit with the lowest possible card, a variance from the procedure is called a "false card." Now the argument runs, suppose West *never* plays a false card. Then his play of the Q the first time makes the probability of the finesse 100%, while if he *always* plays false cards, the finesse will still win 50% of the time. The true probability, then, must be between 100% and 50%, and assuming a random choice of false cards and true cards, the true probability for the finesse must be around 75%. A similar argument can be to analyze the jailer's paradox by considering the extreme situation that the jailer always chooses a name in alphabetical order or always chooses in antialphabetical order.

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