polynomial of A must be x [**1**, Th. 6, p. 204]. Therefore, $A=0$, $a_i=0$ for $i=1,\ldots,n$, and K is formally real. (Note: since K is a principal axis field, the symmetric matrix A is similar to a diagonal matrix over K. Consequently, all the eigenvalues of A in Ω are in K, or $(a_1^2+\cdots+a_n^2)^{1/2}\in K$. We shall later have reason to refer to this fact.)

It remains to be shown that $K_R\subseteq K$, so let $\lambda\in K_R$. Since K is also formally real, λ must be the eigenvalue of some symmetric matrix with entries in K [**4**, Satz 3.3, p. 231]. By hypothesis, such a matrix is similar to a diagonal matrix over K. Therefore, $\lambda\in K$.

We conclude with the observation that if K is a principal axis field, and if A is an n by n symmetric matrix with entries in K, then there exists an orthogonal matrix P (i.e., $P^{-1}=P^t$, the transpose of P) with entries in K such that $P^{-1}AP$ is diagonal. We know that K has to be formally real, so K must be an ordered field [**2**, Cor. 2, p. 274]. An inner product can be defined on K^n, in the usual way, and it will satisfy all the axioms of a real inner product. By the remark at the end of paragraph three in the proof of the theorem, square roots of sums of squares exist in K. Hence, we can define a norm on K^n in the standard way. The Gram-Schmidt process will produce an orthonormal basis of each eigenspace of A, and, since A is similar to a diagonal matrix over K, the union of these bases will provide an orthonormal basis for K^n [**1**, Th. 2, p. 187]. Then P may be chosen as the matrix whose columns are these basis vectors.

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On the Power Sums of the Roots of a Polynomial

Martin L. Brock
2 Old Mamaroneck Rd, #4
White Plains, NY 10605

The sums of the kth powers of the roots of a polynomial are investigated here. Results on these are usually obtained from the theory of symmetric functions (van der Waerden [**3**]) or from Galois theory (Marcus [**2**]). Here, several results are obtained quickly and simply by the use of a theorem of linear algebra. That theorem describes the Jordan form of a matrix.

Let D be an integral domain and Q its quotient field. Let n be in $\mathbf{Z}^+$, f be in $D[x]$, f be monic and $\deg(f)=n$. So

$$f=x^n+a_{n-1}x^{n-1}+\cdots+a_0,$$

for some $a_0,\ldots,a_{n-1}$ in D. Let E be an extension field of the splitting field of f over Q. Let the n roots of f in E, respecting multiplicities, be $r_1,\ldots,r_n$. Let k be in $\mathbf{Z}^+$ and let $s_k(f)$ be the sum of the kth powers of the roots of f. That is,

$$s_k(f) = r_1^k + \cdots + r_n^k.$$

The quantity $s_k(f)$ will be investigated by linear algebra.

Recall the **companion matrix** of f, denoted by C_f. This is the following $n \times n$ matrix over D:

$$C_f = \begin{bmatrix} 0 & 1 & & & \\ & 0 & & \mathbf{0} & \\ & & \ddots & 1 & \\ & \mathbf{0} & & 0 & \\ & & & & 1 \\ -a_0 & & \cdots & -a_{n-2} & -a_{n-1} \end{bmatrix}$$

The reason for introducing this matrix is the following standard (and easily proven) fact (see Herstein [**1**, p. 265]):

$$\text{characteristic polynomial of } C_f = f.$$

Since the characteristic polynomial of C_f (namely, f) splits completely over E (by construction), C_f can be brought to Jordan form by matrices over E (see Herstein [**1**, p. 258]). That is, there is an $n \times n$ invertible matrix, P, over E and an $n \times n$ matrix in Jordan form, J, over E such that

$$PC_fP^{-1} = J.$$

Among other things, J is upper triangular and its main diagonal consists of the roots of the characteristic polynomial, f, with multiplicities respected. That is, the main diagonal consists of $r_1, \dots, r_n$.

Using the last equation, letting k be in $\mathbf{Z}^+$, and taking powers and matrix traces give

$$\begin{aligned} \operatorname{tr}(J^k) &= \operatorname{tr}\left(\left(PC_fP^{-1}\right)^k\right) \\ &= \operatorname{tr}\left(PC_f^kP^{-1}\right) \\ &= \operatorname{tr}\left(C_f^k\right). \end{aligned}$$

The previously mentioned properties about J give

$$\begin{aligned} \operatorname{tr}(J^k) &= r_1^k + \cdots + r_n^k \\ &= s_k(f). \end{aligned}$$

Combining these last two results gives the main result of this note, as stated in the following theorem.

THEOREM. *For k in Z^+, we have*

$$s_k(f) = \operatorname{tr}\left(C_f^k\right).$$

Remarks.

1) Since C_f is immediately gotten from f, this theorem gives a simple way to calculate $s_k(f)$, the sum of the kth powers of the roots of f. The speed of this calculation depends on how fast one can multiply matrices over D. This can be compared with the algorithms offered by the theory of symmetric functions (see van der Waerden [**3**]).
2) Since the entries of C_f are only $0, 1, -a_0, \dots, -a_{n-1}$, this theorem shows at once that $s_k(f)$ is in $\mathbf{Z}[a_0, \dots, a_{n-1}]$, the ring generated by $a_0, \dots, a_{n-1}$. In particular, it shows that $s_k(f)$ is in D. This can be compared with the usual proofs obtained from the theory of symmetric functions (see van der Waerden [**3**]) or Galois theory with the theory of algebraic integers (see

Marcus [2]).

3) The theorem can be used to generate identities valid in *any* ring. For example,

$$r_1^2+\cdots+r_n^2=(r_1+\cdots+r_n)^2-2(r_1r_2+\cdots+r_{n-1}r_n);$$
$$r_1^3+\cdots+r_n^3=(r_1+\cdots+r_n)^3-3(r_1+\cdots+r_n)(r_1r_2+\cdots+r_{n-1}r_n)$$
$$+3(r_1r_2r_3+\cdots+r_{n-2}r_{n-1}r_n).$$

The above results can be extended to the case $k=0$ or k a negative integer, when a_0 is nonzero. This would involve taking the inverse of the companion matrix. (The latter is easily obtained, however, by any of several methods, e.g., matrix adjoints or the Cayley-Hamilton Theorem. Also, it is not too hard to see that this inverse closely resembles the original companion matrix.)

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On the Laplacian

JAMES WARD BROWN

University of Michigan—Dearborn

Dearborn, MI 48128

FRANK FARRIS

Santa Clara University

Santa Clara, CA 95053

In various applied mathematics courses one appearance of the Laplacian operator ∇^2 is in the study of heat distributions. If u is a heat distribution in space, then $\nabla^2 u=0$ if and only if u is a steady-state distribution, one that could be maintained indefinitely inside a box with suitable boundary conditions.

In rectangular coordinates the Laplacian of u is

$$\nabla^2 u=\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}+\frac{\partial^2 u}{\partial z^2}.$$

Careful use of the chain rule gives

$$\nabla^2 u=\frac{\partial^2 u}{\partial r^2}+\frac{1}{r}\frac{\partial u}{\partial r}+\frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2}+\frac{\partial^2 u}{\partial z^2}$$

and

$$\nabla^2 u=\frac{\partial^2 u}{\partial \rho^2}+\frac{2}{\rho}\frac{\partial u}{\partial \rho}+\frac{1}{\rho^2\sin^2\phi}\frac{\partial^2 u}{\partial \theta^2}+\frac{1}{\rho^2}\frac{\partial^2 u}{\partial \phi^2}+\frac{\cot\phi}{\rho^2}\frac{\partial u}{\partial \phi}$$

as the correct formulas for the Laplacian in cylindrical and spherical coordinates [1].