Time-Sharing and the DeMoivre-Laplace Theorem

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One of the most important theorems of statistics is the central limit theorem. Roughly, it states that a sum of (stochastically) independent and identically distributed random variables has an approximately normal distribution, with the approximation improving as the number of random variables increases. A precursor of this theorem is the DeMoivre-Laplace theorem, which states that a sum of independent Bernoulli random variables (with common mean) is approximately normal. The latter theorem was first proved in 1718 by DeMoivre and then generalized by Laplace in 1812. The more general result, known as the central limit theorem, is attributed to Lindeberg (1922). For additional information regarding these theorems and the terms used in this article, the reader may consult any text in mathematical statistics (see, for example, Hoel, Port, and Stone [**1**]).

A Bernoulli random variable is one that takes on only two values, say, 1 with probability p, and 0 with probability $q = 1 - p$. The mean of such a Bernoulli random variable is p, and its variance is given by pq. These variables arise in many settings. For example, the values taken on may be male-female, yes-no, off-on, candidate A-candidate B, and so on. The DeMoivre-Laplace theorem is formally stated below.

DEMOIVRE-LAPLACE THEOREM. *Let $X_1, \dots, X_n$ be a random sample from a Bernoulli population with mean p and let $S_n = X_1 + X_2 + \cdots + X_n$. Then for any real numbers a and b, we have*

$$\lim_{n \to \infty} P\left(a < (S_n - np)/\sqrt{npq} < b\right) = \int_a^b \left(1/\sqrt{2\pi}\right)\exp\left(-z^2/2\right)\, dz.$$

Notice that the random variable $(S_n - np)/\sqrt{npq}$ within the left side of the equation above is formed by *standardizing* the (binomial) random variable S_n, that is, the mean np is subtracted and the resulting expression is divided by the standard deviation $\sqrt{npq}$. On the other hand, the integrand of the right side is the probability density function of a standard normal random variable. Intuitively, the theorem states that a standardized binomial random variable may be approximated by a standard normal random variable.

One of the most common applications of this theorem involves the derivation of approximate confidence intervals to estimate the population mean p. In this note, we will be concerned with a problem which arose in the discussion of time-sharing. Specifically, a large corporation had allotted (rented) 10 computer lines to 40 smaller companies. The corporation later decided to extend this service to encompass 80 customers. Executives of the corporation felt that since the number of customers had doubled, it would be reasonable to double the number of allotted lines to 20. However, computer simulations seemed to indicate that *less* than 20 (actually, 15) lines would suffice. Why? It should be noted that any reduction in the number of lines would save the company an enormous amount of money. We will attempt to model the problem here and show that the DeMoivre-Laplace theorem provides a plausible explanation.

We begin by defining the following random variables:

$$X_i(x) = \begin{cases} 1 & \text{if company } i \text{ is on line at time } x \\ 0 & \text{otherwise.} \end{cases}$$

By definition, each X_i is a Bernoulli random variable. Let p denote the probability that X_i takes on the value 1. For this application, it turns out that each company uses the service about one hour each day. So, we take $p = 1/24$. We let $X = X_1 + X_2 + \cdots + X_{40}$. Notice that we may interpret X as the number of companies that are on line at a particular time. In fact, because the corporation is allotting 10 lines, the probability that a line is open is $\beta = P(X < 10)$. Roughly, β is a measure of customer satisfaction in the sense that the closer β is to 1, the more likely it is that a customer can go on line without waiting. The corporation's goal is to increase the number of lines to accommodate the additional customers without decreasing β.

If we let Z denote the standard normal random variable, the DeMoivre-Laplace theorem tells us that

$$(X - 40p)/\sqrt{40pq} \sim Z \tag{1}$$

or,

$$X \sim \sqrt{40pq}\, Z + 40p,$$

where $\sim$ means "has approximately the same distribution as." We should note that we have made the assumption that the X_i's are independent.

Now, let $W = X_1 + X_2 + \cdots + X_{80}$. If k denotes the number of lines that the company should use for 80 customers and still maintain satisfaction β, then we have the condition

$$P(W < k) = \beta = P(X < 10). \tag{2}$$

Applying the DeMoivre-Laplace theorem to W as we did earlier to X, we obtain

$$W \sim \sqrt{80pq}\, Z + 80p. \tag{3}$$

Combining (1) and (3), we have

$$W \sim \sqrt{80pq}\left[(X - 40p)/\sqrt{40pq}\right] + 80p.$$

Or,

$$W \sim \sqrt{2}\, X + (80 - 40\sqrt{2})\, p. \tag{4}$$

Equations (2) and (4) yield

$$\begin{aligned} P(X < 10) &= P(W < k) \\ &= P\left(\sqrt{2}\, X + (80 - 40\sqrt{2})\, p < k\right) \\ &= P\left(X < \left[k + (40\sqrt{2} - 80)\, p\right]/\sqrt{2}\right). \end{aligned}$$

From these equations, we set

$$10 = \left[k + (40\sqrt{2} - 80)\, p\right]/\sqrt{2}.$$

Solving for k, we obtain

$$k = 10\sqrt{2} + (80 - 40\sqrt{2})\, p.$$

For $p = 1/24 = .04$, we have $k = 15.08$, that is, about 15 lines should provide the same level of satisfaction for 80 customers as 10 lines provide for 40 customers. To see how k varies for selected values of p, we construct the following table.

Usage	Number of lines
p	k
.01	14.38
.04	15.08
.08	16.02
.13	17.19
.17	18.13
.21	19.06
.25	20.00

We see that as the *usage* p for a fixed number of customers increases, the number of lines necessary to maintain customer satisfaction also increases.

References

[1] P. Hoel, S. Port, and C. Stone, Introduction to Probability Theory, Houghton Mifflin Company, 1971, p. 184.

Ladder Approximations of Irrational Numbers

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The approximation of irrational numbers by rationals is a topic which has interested mathematicians since the time of the Greeks. In fact, the Pythagoreans, who are credited with the discovery of $\sqrt{2}$ and its irrationality, had a clever device, known as the system of "side and diameter" numbers, for obtaining successively better rational approximations to $\sqrt{2}$. While the method of side and diameter numbers is thought to have originated before 450 B.C., the earliest account of which we now have record is in the writings of Theon of Smyrna (c. A.D. 140) and may be found in the English translation of Ivor Thomas [**6**, pp. 133–137].

The method of side and diameter numbers is best described algebraically in terms of "Ladder Arithmetic." The ladder consists of a succession of rungs where each rung contains a pair of integers s_k and d_k. The s_k's are associated with the sides and the d_k's with the diameters in the geometric description. For the initial rung, both s_0 and d_0 are 1 and the successive rungs are generated by using the recurrence relations $s_{k+1} = s_k + d_k$ and $d_{k+1} = 2s_k + d_k$ (TABLE 1). Then the quotients d_k/s_k approach the desired limit. The sequence of approximations for $\sqrt{2}$ so

s_k	d_k
1	1
2	3
5	7
12	17
⋮	

TABLE 1

obtained is $1, 3/2, 7/5, \ldots$. The algebraic verification rests on the identity $d_k^2 = 2s_k^2 + (-1)^{k+1}$, which can easily be verified by induction using the recurrence relations given above. Once we have this identity, all that is necessary is to divide through by s_k^2 and take the limit, since the s_k's clearly approach ∞. An interesting observation is that this ladder is closely related to the