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Unevening the Odds of "Even Up"

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The Disclaimer The authors take no responsibility for any gambling hustles or scams based on applications of the principles contained in this note.

The Game "Even Up" is a game of solitaire played with 40 cards from a standard deck that has its jacks, queens, and kings removed. The cards are shuffled and dealt in a row. If a consecutive pair of cards adds to an even number, then that pair can be removed. The object of the game is to remove all of the cards.

More generally, we can play Even Up with $2n$ cards, x of them being odd and $2n-x$ being even. We require the number of cards to be even since the game cannot be won with an odd number of cards. In fact, the game cannot be won when x is odd since odd valued cards are removed in pairs. Harkleroad **[1]** showed that the game involves no skill, in that the outcome is predetermined by the original order of the $2n$ cards, and that the probability of winning is $p(2n,x)=\binom{n}{x/2}^2\Big/\binom{2n}{x}$. Thus the probability of winning the original game is $p(40,20)=0.248$.

A few remarks about $p(2n,x)$ are called for. Clearly $p(2n,0)=1=p(2n,2n)$. By comparing $p(2n,x)$ with $p(2n,x-2)$, one sees that for fixed n the probability of winning is minimized when $x=n$. When n is large, we can use Stirling's formula ($n! \approx (n/e)^n \cdot \sqrt{2\pi n}$) to obtain $p(2n,n) \approx 2/\sqrt{\pi n}$.

For our purposes, any arrangement of $2n$ cards can be represented as the product of a's and b's with a's denoting odd cards and b's denoting even cards. The rules of Even Up reduce to the two multiplications $a^2=1$ and $b^2=1$. Every game simplifies to exactly one string of the form $(ab)^z$, where $-n \le z \le n$ and $(ab)^{-z}=(ba)^z$. Winning games occur when $z=0$. Letting $f(2n,x,z)$ denote the number of arrangements of x a's and $2n-x$ b's that reduce to $(ab)^z$, Harkleroad gives two proofs that

$$f(2n,x,z)=\binom{n}{(x+z)/2}\binom{n}{(x-z)/2}. \tag{1}$$

One proof used induction and the other used a complicated summation. Here we provide a direct combinatorial explanation of equation (1).

The Proof Consider a string of $2n$ symbols as n ordered pairs, where each pair is either (a,a), (b,b), (a,b), or (b,a). Such a string will reduce to $(a,b)^z$, where z is the number of (a,b) pairs minus the number of (b,a) pairs. If we let the four pairs have scores of $0,0,1$, and -1 respectively, then z denotes the total score. Another way to calculate the score is to assign a value to each symbol as follows: a beginning a in an ordered pair gets a score of 1, an ending a in an ordered pair gets a score of -1, and all b's get a score of 0. Hence the total score is equal to the number of beginning a's minus the number of ending a's. If x' denotes the number of beginning a's $(0\le x'\le x)$, then the score $z=x'-(x-x')=2x'-x$. Thus to achieve a score of z, we must choose $x'=(x+z)/2$ of our n pairs to begin with an a and $x-x'=(x-z)/2$ of our n pairs to end with an a. This completes the proof of equation (1).

The probability that a game with x a's and $2n-x$ b's reduces to $(ab)^z$ is $f(2n,x,z)/\binom{2n}{x}$. Notice that $z=2x'-x$ will always have the same parity as x. (This is reflected in (1) since $\binom{n}{k/2}$ is 0 when k is odd.) Thus, we see again that a game with an *even* number of cards but *odd* number of odd cards is impossible to win.

The Scam After explaining the game of Even Up to your mark, challenge him to a duel. Wager that you can win the game in strictly fewer attempts than he can. He can shuffle your cards before every deal. You play the game until you win. Say it takes you 4 attempts. Now it's his turn to play. How can you be sure that it will take him at least 5 attempts? When *you* play the game, use 10 even cards and 10 odd cards. Each attempt has a 34% chance of success. After you win, shuffle the cards, but secretly add (or remove, if you prefer) one even and one odd card to his deck. This will make his winning probability 0.

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