Math Bite: On the Definition of Collineation

Let V and W be vector spaces over some field $\mathbb{F}$, and suppose that $\dim V \geq 2$. A *collineation* is a one-to-one function $f: V \to W$ such that if $x, y, z \in V$ are collinear, then $f(x), f(y), f(z)$ are collinear. The definition says, in essence, that a collineation sends lines to lines. But this is not the precise content of the definition. For one thing, the definition requires that a collineation be one-to-one, which is apparently stronger than simply preserving collinearity. For another, at least on casual inspection, the definition does not require the image of a line to be a line. It would suffice that the image of a line be a *subset* of a line—although this turns out not to occur. (Indeed, one version of the Fundamental Theorem of Projective Geometry states that, for $\mathbb{F} = \mathbb{R}$, every collineation is an affine map.) Given these observations, it is natural to wonder just how far the idea of sending lines to lines goes towards characterizing collineations. The answer is: *it almost does.*

THEOREM. *Suppose that $f: V \to W$ is a function for which the image of every line in V is a line in W. Then either f is a collineation or $f(V)$ has dimension 1.*

Proof. First observe that f sends (affine) subspaces to (affine) subspaces. Indeed, if A is a subspace of V and if $x, y \in f(A)$ are distinct points, then $x = f(a)$ and $y = f(b)$ for some distinct $a, b \in A$. Then if L is the unique line through a and b, we have $L \subseteq A$, so the line $f(L)$ through x and y is contained in $f(A)$.

Suppose that $\dim f(V) > 1$ and that $x, y \in V$ with $f(x) = f(y)$. Let L_{xy} be the line through x and y. By hypothesis, there exists z with $f(z) \notin f(L_{xy})$. Let A be a plane through x, y, and z. Let L be the line through $f(x)$ and $f(z)$, and let L_{xz} (resp. L_{yz}) be the line through x and z (resp. y and z). So $f(L_{xz}) = f(L_{yz}) = L$. Note that dim $f(A) > 1$, since $f(A)$ contains the plane through $f(L_{xy})$ and $f(z)$. Choose any line L' in $f(A)$ that is disjoint from L. Choose a line $\hat{L}$ in A with $f(\hat{L}) = L'$. Since L and L' are disjoint, L_{xz} and $\hat{L}$ must be disjoint. Hence, since they both belong to A, L_{xz} and $\hat{L}$ are parallel. Similarly, L_{yz} and $\hat{L}$ are parallel. Hence L_{xz} and L_{yz} are parallel. Since z belongs to both lines, $L_{xz} = L_{yz}$. So we have two distinct lines, L_{xz} and L_{xy}, both of which contain x and y. Hence $x = y$, and the proof is complete.

In fact, there do exist functions that send lines to lines but are not collineations. Suppose that V is a real normed vector space with norm $\|\cdot\|$. Consider $f: V \to \mathbb{R}$, $f(x) = \|x\| \cdot \sin(\|x\|)$. Since the codomain has dimension 1, the only thing that needs verifying here is that the image of every line is the entire codomain, and not just part of it.

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