Bold Play Is Best: A Simple Proof

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Introduction In the classical ruin problem, a gambler with initial capital i dollars plays against an adversary with initial capital $a-i$ dollars. Here i and a are positive integers, $i<a$. The gambler wins a dollar with probability p and loses a dollar with probability $q=1-p$; the game is repeated until one of the players goes broke. (Part of the analysis of the problem shows the game cannot go on forever.) Another interpretation of this model is that of a gambler playing a game in a casino (the adversary) where the gambler plays until she goes broke or until she wins a fixed predetermined amount, at which time she quits. Given a total fixed capital a, we are interested in the probability q_i that the gambler starting with initial capital i, $1 \le i \le a-1$, is ruined.

This problem is often discussed in a first course in probability and introduces the student to the ideas of random walks and Markov chains. Our main reference is the classic book of Feller [**3**, chapter 14]; see also [**1**], [**4**], and [**5**]. We will regard the sequence of the gambler's fortune after each play as a random walk in the interval $[0, a]$, with absorbing barriers at 0 and a. The probability of ruin is the probability of hitting 0 before hitting a.

Using a difference equation approach and some algebra, the following solution is derived in the case $p \ne 1/2$ (see, e.g., [**3**]):

$$q_i = \frac{\left(\frac{q}{p}\right)^a - \left(\frac{q}{p}\right)^i}{\left(\frac{q}{p}\right)^a - 1}. \tag{1}$$

This is valid for $1 \le i \le a-1$, and even for the boundary values 0 and a if the intuitively reasonable definitions $q_0 = 1$, $q_a = 0$ are made. Thus a gambler with 100 dollars in her pocket, playing repeated games of craps ($p \approx .493$) and determined to win 10 dollars or go broke in the attempt, will be ruined with approximate probability .253.

Now what happens if the gambler changes the stakes: instead of betting a dollar at each play, she bets a fixed s dollars? Here s must be possible in the sense that both i and a must be divisible by s. For example, the gambler with 100 dollars who wants to win 10 dollars can bet at 1, 2, 5, or 10 dollar stakes. What does changing the stakes do to the gambler's probability of ruin? What is the gambler's best strategy to minimize the probability of ruin?

From the random walk point of view, changing the stakes means the unit of fortune has changed, so the gambler with 100 dollars determined to win 10 dollars betting at 10 dollar stakes should have the same probability of ruin as the gambler with 10 dollars planning to quit after winning a dollar, and playing at one dollar stakes. Define $q_i(s)$ as the probability of the gambler's ruin, given that she starts with i dollars and bets at fixed s dollar stakes. For ease of notation, set $r = \frac{q}{p}$ in (1) and then note that this relation, together with the above remarks on changing stakes, shows

$$q_i(s) = \frac{r^{\frac{a}{s}} - r^{\frac{i}{s}}}{r^{\frac{a}{s}} - 1}. \tag{2}$$

Let us assume from now on that *the gambler plays an unfavorable game, that is,* $p < 1/2$. This, after all, is the typical situation for a gambler at a casino. It turns out, surprisingly to many, that the gambler minimizes her ruin probability by playing *boldly*, that is, by playing at the highest stakes possible. Feller **[3]** states this fact about bold play, but only proves it in the case of doubling or halving bets. Chung **[1]** and Ross **[5]** discuss the gambler's ruin problem, but omit any mention of what happens when the stakes are changed. In my own book **[4]** the result is stated but not proved. At a more advanced perspective, Dubins and Savage **[2]** prove the optimality of bold play in the unfavorable game case in a much more general context than the one considered here, but their proofs require a lot of high-powered mathematical machinery. Yet a simple proof of this interesting fact in the classical case outlined above, although not completely trivial, is short and depends only on some elementary calculus. Here's how it goes.

The result Consider the right-hand side of equation (2) as a continuous function of s for fixed r, i, and a, and $s > 0$. If this function can be proved to be decreasing in s, the optimality of bold play follows. The obvious approach is to differentiate the right-hand side of (2) with respect to s and see that the result is negative. But the obvious approach gives a rather messy expression, not immediately seen to be negative. Instead, we take the derivative in two steps, and use the chain rule to arrive at the conclusion.

THEOREM. *If* $p < 1/2$, *the function*

$$q_i(s) = \frac{r^{\frac{a}{s}} - r^{\frac{i}{s}}}{r^{\frac{a}{s}} - 1}$$

is a decreasing function of s *on* $s > 0$.

Proof. Let $w = r^{\frac{i}{s}}$. Since $p < 1/2$ it follows that $w > 1$. Let $ki = a$. Since $i < a$ we must have $k > 1$. Now rewrite $q_i(s)$ as

$$q_i(s) = \frac{w^k - w}{w^k - 1} = 1 + \frac{1 - w}{w^k - 1}. \tag{3}$$

Consider the derivative $dq_i(s)/dw$; we will show this derivative is positive on $w > 1$. This derivative can be found by taking the derivative of the second term of the right-hand side of (3). Using this term and the quotient rule for taking derivatives, the sign of $dq_i(s)/dw$ is seen to be determined by the quantity

$$-(w^k - 1) - (1 - w) \cdot kw^{k-1}. \tag{4}$$

We claim that this expression is positive for all $w > 1$. To see this, observe that the positivity of (4) is equivalent to the inequality

$$h(w) := (k-1)w^k - kw^{k-1} > -1. \tag{5}$$

Consider the function $h(w)$ on the interval $w > 1$. We get

$$h'(w) = k(k-1)(w^{k-1} - w^{k-2}),$$

which is positive because $w > 1$ and $k > 1$. Thus $h(w)$ is an increasing function on $w > 1$. But note that $h(1) = -1$; this observation completes the proof of the inequality (5). Consequently, the derivative $dq_i(s)/dw$ is positive on $w > 1$. Moreover, from the definition of w it is clear that w is a decreasing function of s, that is $dw/ds < 0$, so that

$$\frac{dq_i(s)}{ds} = \frac{dq_i(s)}{dw} \cdot \frac{dw}{ds} < 0,$$

and the proof of the theorem is complete.

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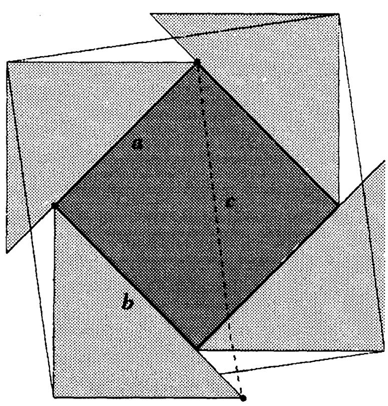
Proof Without Words: $a^2 + b^2 = c^2$



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