It is interesting to repeat these observations, using higher and higher precisions in the **NDSolve** routine. The higher the precision, the longer the trajectory remains "stuck," as it should. It is also instructive to experiment with initial values different from but close to $(0, 1/\sqrt{2})$. If one takes the initial values $x(0) = 0$ and $\dot{x}(0) = 1/\sqrt{2} - 10^{-6}$, the solution is oscillatory and a plot of it very closely resembles a square wave. It is a good student exercise to predict the motion for starting points on other portions of the separatrix. The separatrix actually consists of six separate trajectories and two stationary (saddle) points.

We recommend the interesting paper **[3]** for a more in-depth discussion of precision problems and how to predict over what time interval one can expect a numerical solution to be accurate.

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A Bubble Theorem

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Introduction It is always a good practice to provide the physical content of an analytical result. The following algebraic inequality lends itself well to this purpose: For any finite sequence of real numbers $r_1, r_2, \ldots, r_N$, we have

$$\left(r_1^3 + r_2^3 + \cdots + r_N^3\right)^2 \le \left(r_1^2 + r_2^2 + \cdots + r_N^2\right)^3. \qquad (1)$$

A standard proof is given in **[1]**. An alternative proof follows from the isoperimetric inequality

$$A^3 \ge 36\pi V^2,$$

where A is the surface area and V the volume of any three-dimensional body. Setting the area $A = \sum_{i=1}^{N} 4\pi r^2$ and the volume $V = \sum_{i=1}^{N} (4/3)\pi r^3$ yields (1).

A bubble proof We give yet another proof, now using elements of surface tension theory and ideal gas laws to the formation and coalescence of bubbles. This proof, found in **[2]**, runs as follows.

According to a well-known result in surface tension theory, when a spherical bubble of radius R is formed in the air, there is a difference of pressure between the inside and the outside of the surface film given by

$$p = p_0 + \frac{2T}{R}, \tag{2}$$

where p_0 is the (external) atmospheric pressure on the surface film of the bubble, p is the internal pressure, and T is the surface tension that maintains the bubble [**3**].

Suppose initially that N spherical bubbles of radii $R_1, R_2, \dots, R_N$ float in the air under the same surface tension T and internal pressures $p_1, p_2, \dots, p_N$. According to (2),

$$p_k = p_0 + \frac{2T}{R_k}, \quad k = 1, 2, \dots N. \tag{3}$$

Now suppose that all N bubbles come close enough to be drawn together by surface tension and combine to form a single spherical bubble of radius R and internal pressure p, also obeying equation (2). When this happens, the product of the internal pressure p and the volume v of the resulting bubble formed by the coalescence of the initial bubbles is, according to the ideal gas law [**3**], given by

$$pv = p_1 v_1 + \cdots + p_N v_N, \tag{4}$$

where v_k $(k = 1, 2, \dots, N)$ are the volumes of the individual bubbles before the coalescence took place. For spherical bubbles, (4) becomes

$$pR^3 = p_1 R_1^3 + \cdots + p_N R_N^3. \tag{5}$$

Substituting the values of p and p_k given in (2) and (3) into (5), we obtain

$$R^3 - R_1^3 - R_2^3 - \cdots - R_N^3 = \frac{2T}{p_0}\left(R_1^2 + R_2^2 + \cdots + R_N^2 - R^2\right). \tag{6}$$

Now, if the total amount of air in the bubbles does not change, the surface area of the resulting bubble formed by the coalescence of the bubbles is always smaller than the sum of the surface area of the individual bubbles before coalescence. Thus,

$$R_1^2 + R_2^2 + \cdots + R_N^2 \geq R^2. \tag{7}$$

Since the potential energy of a bubble is proportional to its surface area, (7) is a physical condition that the surface energy of the system is minimal after the coalescence.

It follows from (7) and the fact that p_0 and T are positive constants that the left hand side of equation (6) satisfies

$$R_1^3 + R_2^3 + \cdots + R_N^3 \leq R^3. \tag{8}$$

The equality, which implies conservation of volumes, holds when the excess pressure in the bubble film is much less the atmospheric pressure. Combining (7) and (8) yields the inequality (1), which is also valid for negative numbers.

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