Flett's Mean Value Theorem for Holomorphic Functions

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The mean value theorem is a well known result usually covered in a first semester calculus course. There are many other types of mean value theorems that are less well known. In 1958, T. M. Flett **[3]** proved one such result. Specifically, if $f:[a,b]\to\mathbb{R}$ is differentiable and satisfies $f'(a)=f'(b)$, then there exists η in the open interval (a,b) such that $f(\eta)-f(a)=(\eta-a)f'(\eta)$. FIGURE 1 shows the nice geometric interpretation of this result. If the curve $y=f(x)$ has a tangent at each point in $[a,b]$, and if the tangents at $(a,f(a))$ and $(b,f(b))$ are parallel, then there is an intermediate point η such that the tangent at $(\eta,f(\eta))$ passes through the point $(a,f(a))$. For other examples of mean value theorems we refer the reader to **[4]**–**[9]**.

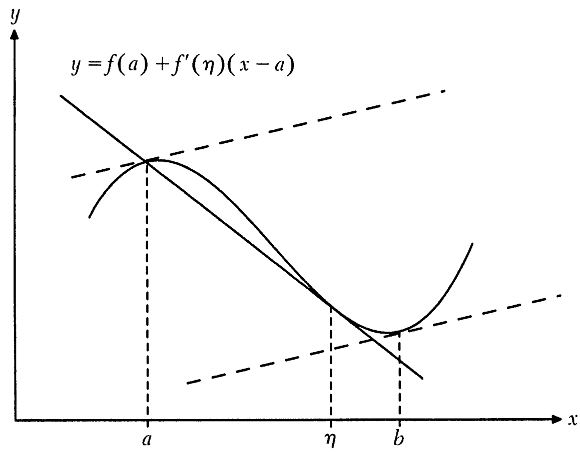

FIGURE 1
Geometric interpretation of Flett's theorem.

These results do not, in general, immediately extend to holomorphic functions of a complex variable. For the case of Rolle's theorem, the function $f(z)=e^z-1$ has value 0 at $z=0$ and $z=2\pi i$, but $f'(z)=e^z$ has no zeros in the complex plane. Evard and Jafari **[1]** get around this difficulty by working with the real and imaginary parts of a holomorphic function. Another approach is taken by Samuelsson in **[8]**. The goal of this note is to prove a version of Flett's theorem for holomorphic functions of a complex variable in the spirit of Evard and Jafari.

The first step is to extend Flett's mean value theorem for real functions to a result that does not depend on the hypothesis $f'(a)=f'(b)$, but reduces to Flett's theorem when this is the case.

THEOREM 1. *If $f:[a,b]\to\mathbb{R}$ is a differentiable function, then there exists a point $\eta\in(a,b)$ such that*

$$f(\eta)-f(a)=(\eta-a)f'(\eta)-\frac{1}{2}\frac{f'(b)-f'(a)}{b-a}(\eta-a)^2.$$

Proof. Consider the auxiliary function $\psi:[a,b]\to\mathbb{R}$ defined by

$$\psi(x)=f(x)-\frac{1}{2}\frac{f'(b)-f'(a)}{b-a}(x-a)^2.$$

Then ψ is differentiable on $[a,b]$, and

$$\psi'(x)=f'(x)-\frac{f'(b)-f'(a)}{b-a}(x-a).$$

It follows that $\psi'(a)=\psi'(b)=f'(a)$. Applying Flett's mean value theorem to ψ gives $\psi(\eta)-\psi(a)=(\eta-a)\psi'(\eta)$ for some $\eta\in(a,b)$. Rewriting ψ and ψ' in terms of f gives the asserted result.

Our next step is to introduce some notation. Let $\mathbb{C}$ denote the set of complex numbers. For distinct a and b in $\mathbb{C}$, let $[a,b]$ denote the set $\{a+t(b-a)|t\in[0,1]\}$; we will refer to $[a,b]$ as a *line segment* or a *closed interval* in $\mathbb{C}$. Similarly, (a,b) denotes the set $\{a+t(b-a)|t\in(0,1)\}$.

Flett's theorem is not valid for complex valued functions of a complex variable. To see this, consider the function $f(z)=e^z-z$. Then f is holomorphic, and $f'(z)=e^z-1$. Therefore, we have $f'(2k\pi i)=e^{2k\pi i}-1=0$ for all integers k. In particular, $f'(0)=f'(2\pi i)$, that is, the derivatives of f at the endpoints of the closed interval $[0,2\pi i]$ are equal. Nevertheless, the equation

$$f(z)-f(0)=f'(z)z$$

has no solution on the interval $(0,2\pi i)$, as we now show. The equation above gives $(1-z)=e^{-z}$ and, since $z=iy$, we get $1-iy=\cos y-i\sin y$. The comparison of the real and imaginary parts gives the system $\cos y=1$ and $\sin y=y$, which has no solution in the interval $(0,2\pi)$. Thus Flett's theorem fails in the complex domain.

We now present a generalization of Theorem 1 for holomorphic functions where, for any two complex numbers u and v,

$$\langle u,v\rangle=\mathrm{Re}(u\bar{v}).$$

THEOREM 2. *Let f be a holomorphic function defined on an open convex subset D of $\mathbb{C}$. Let a and b be two distinct points in D. Then there exist $z_1, z_2\in(a,b)$ such that*

$$\mathrm{Re}(f'(z_1))=\frac{\langle b-a,f(z_1)-f(a)\rangle}{\langle b-a,z_1-a\rangle}+\frac{1}{2}\frac{\mathrm{Re}(f'(b)-f'(a))}{b-a}(z_1-a)$$

and

$$\mathrm{Im}(f'(z_2))=\frac{\langle b-a,-i[f(z_2)-f(a)]\rangle}{\langle b-a,z_2-a\rangle}+\frac{1}{2}\frac{\mathrm{Im}(f'(b)-f'(a))}{b-a}(z_2-a).$$

Proof. Let $u(z) = \mathrm{Re}(f(z))$ and $v(z) = \mathrm{Im}(f(z))$ for $z \in D$. We now define the auxiliary function $\phi:[0,1] \to \mathbb{R}$ by

$$\phi(t) = \langle b-a, f(a+t(b-a))\rangle, \tag{1}$$

which is

$$\phi(t) = \mathrm{Re}(b-a)u(a+t(b-a)) + \mathrm{Im}(b-a)v(a+t(b-a))$$

for every $t \in [0,1]$. Therefore, using the Cauchy–Riemann equations, we get

$$\begin{aligned}\phi'(t) &= \langle b-a, (b-a)f'(a+t(b-a))\rangle \\ &= \mathrm{Re}\big((b-a)^2\big)\frac{\partial u(z)}{\partial x} + \mathrm{Im}\big((b-a)^2\big)\frac{\partial u(z)}{\partial x} \\ &= |b-a|^2\frac{\partial u(z)}{\partial x} \\ &= |b-a|^2\,\mathrm{Re}(f'(z)).\end{aligned}$$

Applying Theorem 1 to ϕ on $[0,1]$, we obtain

$$t_1\phi'(t_1) = \phi(t_1) - \phi(0) + \frac{1}{2}\frac{\phi'(1)-\phi'(0)}{1-0}(t_1-0)^2$$

for some $t_1 \in (0,1)$. Thus

$$t_1|b-a|^2\,\mathrm{Re}(f'(z_1)) = \phi(t_1) - \phi(0) + \frac{1}{2}[\phi'(1)-\phi'(0)]t_1^2,$$

where $z_1 = a + t_1(b-a)$. Further, since $z_1 = a + t_1(b-a)$ and $t_1 \in [0,1]$, we have $t_1|b-a|^2 = \langle b-a, z_1-a\rangle$. Hence the equation $t_1|b-a|^2\,\mathrm{Re}(f'(z_1)) = \phi(t_1) - \phi(0) + \frac{1}{2}[\phi'(1)-\phi'(0)]t_1^2$ reduces to

$$\mathrm{Re}(f'(z_1)) = \frac{\phi(t_1)-\phi(0)}{t_1|b-a|^2} + \frac{1}{2}\frac{\phi'(1)-\phi'(0)}{|b-a|^2}t_1.$$

Using (1) and the fact that $z_1 = a + t_1(b-a)$ in the above equation, we obtain

$$\mathrm{Re}(f'(z_1)) = \frac{\langle b-a, f(z_1)-f(a)\rangle}{\langle b-a, z_1-a\rangle} + \frac{1}{2}\frac{\mathrm{Re}(f'(b)-f'(a))}{b-a}(z_1-a). \tag{2}$$

Letting $g = -if$, we have

$$\mathrm{Re}(g'(z)) = \frac{\partial v(z)}{\partial x} = -\frac{\partial u(z)}{\partial y} = \mathrm{Im}(f'(z)).$$

Now, applying the first part to g, we obtain

$$\mathrm{Re}(g'(z_2)) = \frac{\langle b-a, g(z_2)-g(a)\rangle}{\langle b-a, z_2-a\rangle} + \frac{1}{2}\frac{\mathrm{Re}(g'(b)-g'(a))}{b-a}(z_2-a)$$

for some $z_2 \in (a,b)$. By (2) the above yields

$$\mathrm{Im}(f'(z_2)) = \frac{\langle b-a, -i[f(z_2)-f(a)]\rangle}{\langle b-a, z_2-a\rangle} + \frac{1}{2}\frac{\mathrm{Im}(f'(b)-f'(a))}{b-a}(z_2-a)$$

and the proof is complete.

The next corollary follows immediately; it is the complex version of Flett's mean value theorem.

COROLLARY. *Let f be a holomorphic function defined on an open convex subset D of $\mathbb{C}$. Let a and b be two distinct points in D, and $f'(a)=f'(b)$. Then there exist $z_1, z_2 \in (a,b)$ such that*

$$\mathrm{Re}(f'(z_1)) = \frac{\langle b-a, f(z_1)-f(a)\rangle}{\langle b-a, z_1-a\rangle}$$

and

$$\mathrm{Im}(f'(z_2)) = \frac{\langle b-a, -i[f(z_2)-f(a)]\rangle}{\langle b-a, z_2-a\rangle}.$$

Returning to our original example $f(z)=e^z-z$, z_1 and z_2 predicted by Theorem 2 have values $z_1 \approx 4.49341i$ and $z_2 \approx 2.33112i$.

There are many ways to generalize the results of this note. For example, in 1966, Trahan **[9]** extended Flett's theorem by replacing the boundary condition $f'(a)=f'(b)$ with $[f'(a)-m][f'(b)-m]>0$, where $m=\frac{f(b)-f(a)}{b-a}$. A similar modification of the Corollary gives an extension of Trahan's result to the complex plane. Specifically, replace the boundary condition $f'(a)=f'(b)$ with the conditions $[\mathrm{Re}(f'(a))-m_1][\mathrm{Re}(f'(b))-m_1]>0$ and $[\mathrm{Im}(f'(a))-m_2][\mathrm{Im}(f'(b))-m_2]>0$, where $m_1=\frac{\langle b-a, f(b)-f(a)\rangle}{\langle b-a, b-a\rangle}$ and $m_2=\frac{\langle b-a, -i[f(b)-f(a)]\rangle}{\langle b-a, b-a\rangle}$. In 1995, Evard, Jafari, and Polyakov **[2]** extended Rolle's theorem for holomorphic functions on line segments **[1]** to a result satisfied by arbitrary curves connecting a and b where $f(a)=f(b)$. For their result to hold the domain must satisfy a relatively weak *almost convexity* condition. The same method could be used here, but we wish to limit ourselves to a more classical setting.

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