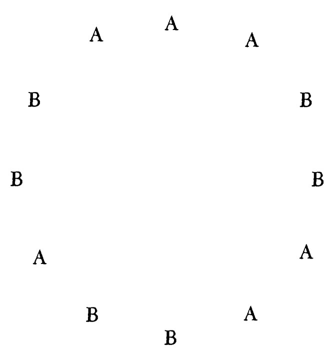
The Hexachordal Theorem: A Mathematical Look at Interval Relations in Twelve-Tone Composition

STEVEN K. BLAU
Physics Department
Ripon College
Ripon, WI 54971

Introduction Take the numbers 1 through 12 and divide them into two complementary six-member subsets A and B. Each subset has $\binom{6}{2} = 15$ pairs of members; the *interval* between any pair of numbers is the (positive) difference between them, subject to the equivalence $n \sim 12 - n$. So, for the pair $(1, 11)$ the interval may be said to be either 10 or 2. A convenient way to represent a complementary pair of interval sets, which makes visually apparent the equivalence just described, is to present a "clock face" with six A's and six B's, as follows:



In the case shown, $A = \{1, 4, 5, 8, 11, 12\}$ and $B = \{2, 3, 6, 7, 9, 10\}$. The interval between the A's at one o'clock and eleven o'clock is either 10 (counting clockwise) or 2 (counting anticlockwise). In what follows I will use the equivalence relation to describe all intervals as being between 1 and 6.

The *interval multiset* associated with a six-member set is the collection of 15 intervals determined by all possible pairs drawn from the set. For the set A shown, the interval multiset is $\{1, 1, 1, 2, 3, 3, 3, 4, 4, 4, 4, 5, 5, 5, 6\}$. As one can check directly, the set B has the *same* interval multiset. This is no accident—the equality of the interval multisets is the content of the hexachordal theorem.

THEOREM. *Let the numbers* 1 *through* 12 *be partitioned into any two complementary sets A and B, each with six elements. Then A and B have identical interval multisets.*

Before proving the theorem, we consider its musical meaning.

The musical context The hexachordal theorem was empirically discovered by composers working with Arnold Schönberg's twelve-tone method. In the musical context, the numbers 1 through 12 represent the twelve notes of the chromatic scale. Twelve-tone composers use a twelve note "row" containing all the members of the chromatic scale, arranged as they see fit, as a basis for their melodic composition. They often think of the first and second halves of the row as two complementary *hexachords*. In the musical context, "complementary" means that a note contained in one hexachord is not present in its companion. In the mathematical setting, complementary hexachords are complementary six-member subsets of $\{1, 2,, \ldots, 12\}$.

What I called an interval in the theorem would also be called an interval by composers: it represents the number of semitones (notes on a keyboard) needed to connect one note to another. Thus the hexachordal theorem is a statement about the interval structures of the two complementary halves of a twelve-tone row.

The hexachordal theorem was first proved by Milton Babbitt, a celebrated composer with a degree in mathematics, and David Lewin, then a graduate student in mathematics. In describing his proof with Lewin, Babbitt wrote, "We used topological methods. We hit this little problem with all kinds of heavy hammers, and we solved it" **[1]**. Later, Lewin (working on his own) and Ralph Fox constructed different proofs using group-theoretic methods. Lewin described his work on the hexachordal and related theorems in the *Journal of Music Theory* (see **[2]** and **[3]**). He did not include a proof of the hexachordal theorem, but sketched his work on related theorems, observing after one of his proofs **[2]**: "The mathematical reasoning by which I arrived at this result is not communicable to a reader who does not have considerable mathematical training." The proof of the hexachordal theorem that follows requires no advanced mathematics; it can be followed by musicians with little mathematical training, and it may interest mathematicians.

Challenged to give a four-word history of western music up to 1900, I would offer: "Modulations became more frequent." By the late 1800s in the works of Wagner, for example, there are sections where the modulations come so quickly that tonalities are established only for seconds before they change. Beethoven, by contrast, typically allowed tonalities to be established for a much greater time period before modulating to a new tonality. Early in his career, Arnold Schönberg wrote music following the Wagnerian line, but later decided that such music contained a sort of inconsistency. The momentum of music created since Bach was pointed toward an equality of the notes of the chromatic scale, but early twentieth-century composers still felt tied to the tonal conventions of earlier centuries. Schönberg advocated an "emancipation of dissonance," abandoning the notion that music must be conceived in terms of tonalities. A corollary was the freedom to use all twelve notes of the chromatic scale equally in composing, though Schönberg did occasionally write tonal music throughout his career.

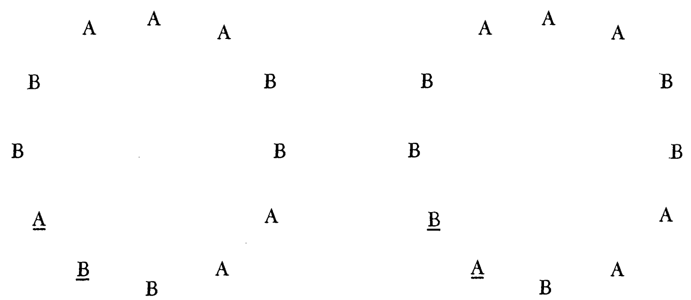
The early pieces of Schönberg and his "Second Viennese School" were almost all short and aphoristic. The freedom enjoyed by composers was apparently so great that it was difficult to write on a grand scale music that maintained its internal logic. Schönberg sought a structure that would allow atonal composers to create coherent works on a large scale. His solution was the "method of composing with twelve tones related only to one another." One begins with a scale, or row, containing the twelve notes of the chromatic scale arranged in some fixed order. Then one manipulates the row in prescribed ways in composing. Part of the composer's skill is knowing when to break Schönberg's rules, but these rules do offer cohesion to atonal compositions. Some twelve-tone composers felt that their work would be further unified if the interval multisets implied by various subsets of their twelve-tone rows were identical.

Consequently, they empirically discovered the hexachordal theorem, as an offshoot of their search for unification. Babbitt and Lewin, and then others, proved the empirical result mathematically.

Proof of the theorem Given any pair of complementary hexachords A and B, displayed as a clock face, one can generate a new complementary pair by switching an A with a neighboring B. Indeed, *any* complementary pair of hexachords can be generated, through a sequence of switches, from the pairing with A's in positions 1–6 and B's in positions 7–12. This special pairing clearly satisfies the conclusion of the hexachordal theorem. Therefore, the hexachordal theorem is a consequence of the following lemma.

LEMMA. *Suppose that two complementary hexachords A and B have identical interval multisets. Then the hexachords obtained by switching a pair of neighboring A and B elements also have identical interval multisets.*

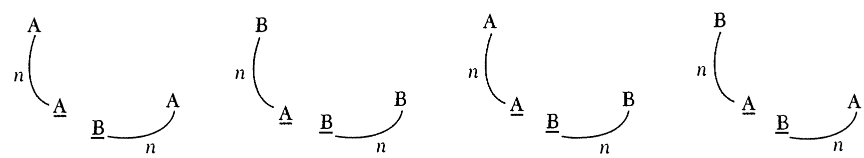
Proof. FIGURE 1 shows, at left, the clock face introduced earlier to represent a particular pair of hexachords. Switching the underlined entries (an A at eight o'clock and a B at seven o'clock) produces the the clock face on the right:



Before the switch After the switch

FIGURE 1
Switching an adjacent pair.

After identifying an adjacent pair of A and B elements to be switched, one may partition the remaining 10 elements into 5 pairs: for each integer n from 1 to 5, we consider the pair of elements lying n hours to either side of the switched A and B. There are four possibilities for the membership of each such pairs: two A's, two B's, or one of each (in either order). All four possibilities are illustrated in FIGURE 2 (the particular values of n correspond to the example illustrated above):



$n = 3$ $n = 1$ $n = 4$ $n = 2$

FIGURE 2
Pairs of elements lying n hours to either side of the switched A and B.

By considering the five pairs defined as above, one considers all the intervals possibly affected by the switch.

We consider four cases. If a pair contains two A's (as illustrated for $n = 3$), then the switch changes two intervals in the A hexachord: two original intervals, with lengths n and $n + 1$, become two new intervals, with lengths $n + 1$ and n, respectively. Hence the switch has no net effect on the A hexachord multiset (or on the B multiset, since the pair involves no B's). A similar argument shows that the switch has no effect if both elements in a pair are B's (as illustrated for $n = 1$).

If the element of the pair nearest to the switched A is an A, and the other element a B (as shown in FIGURE 2 for $n = 4$), then the switch does alter the interval multisets for the A and B hexachords, but in identical ways—in each multiset, an original interval of size n becomes a new interval of size $n + 1$. The remaining case (illustrated for $n = 2$) is similar.

Thus, for all n, the switching operation has the same effect, if any, on both the A and the B interval multisets, and the proof is complete.

A generalization Nothing in the proof above relied on the (musical) facts that the chromatic scale has 12 notes and that composers considered dividing the chromatic scale into two hexachords of equal length. The following theorem can be proved by the same method, and was expressed in a different form by Lewin [3] for the case $N = 12$.

THEOREM. *Let the set* $\{1, 2, \dots, N\}$ *be partitioned into disjoint sets A and B, with size a and* $N - a$*, respectively. Define interval multisets for the A chord and the B chord as was done for the hexachordal theorem. Let* $A(i)$ *be the number of i's in the A interval multiset, and similarly for* $B(i)$*. Consider the special partition* $A_0 = \{1, 2, 3, \dots, a\}$, $B_0 = \{a + 1, a + 2, a + 3, \dots, N\}$*. Then, for all i,* $A(i) - B(i) = A_0(i) - B_0(i)$.

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