

Math Bite: Further Generalizations of a Curiosity that Feynman Remembered All His Life

Richard Feynman remembered that, when a boy, he was told by Morrie Jacobs [3, page 47] that $\cos 20^\circ \times \cos 40^\circ \times \cos 80^\circ = 1/8$. Beyer, Louck, and Zeilberger pointed out [1] that “Morrie Jacobs’ identity is the special case $k = 3$, $a = 20^\circ$ of the following identity that follows by induction on k using $\sin 2b = 2 \sin b \cos b$, with $b = 2^{k-1}a$ ”:

$$2^k \prod_{j=0}^{k-1} \cos(2^j a) = \frac{\sin(2^k a)}{\sin a}.$$

Morrie Jacobs’ identity can be generalized in various other ways, several of which are given in the splendid old textbook [2] by Durell & Robson. In particular, Jacobs’ identity is the special case $m = 4$ of the first of the 4 identities in positive integers m :

$$\prod_{r=1}^m \cos\left(\frac{r\pi}{2m+1}\right) = 2^{-m}, \quad \prod_{r=1}^m \cos\left(\frac{(2r-1)\pi}{4m}\right) = 2^{1-m} \sqrt{m},$$

[2, page 225, Ex. 24], which transform simply to give:

$$\prod_{r=1}^m \sin\left(\frac{r\pi}{2m+1}\right) = 2^{-m}, \quad \prod_{r=1}^m \sin\left(\frac{(2r-1)\pi}{4m}\right) = 2^{1-m} \sqrt{m},$$

[2, page 225, Ex. 25].

Dividing the products of sines by the corresponding products of cosines, we get trivial identities for tangents: $\prod_{r=1}^m \tan\left(\frac{r\pi}{2m+1}\right) = 1$, and [2, page 225, Ex.30]

$$\prod_{r=1}^m \tan\left(\frac{(2r-1)\pi}{4m}\right) = 1.$$

There are various other generalizations, including the following identity [2, page 225, Ex. 29] in positive integers n :

$$\prod_{r=1}^n \cos\left(\frac{(2r-1)\pi}{2n}\right) = 2^{1-n} \cos\left(\frac{n\pi}{2}\right).$$

For odd n the product equals 0, but for even $n = 2m$, this becomes the identity

$$\prod_{r=1}^{2m} \cos\left(\frac{(2r-1)\pi}{4m}\right) = \frac{2}{(-4)^m}.$$

Also [2, page 226, Ex. 31]:

$$\prod_{r=1}^{2m+1} \cos\left(\frac{(4r-1)\pi}{8m+4}\right) = \frac{-1}{\sqrt{2}(-4)^m}, \quad \prod_{r=1}^{2m} \cos\left(\frac{(4r-1)\pi}{8m}\right) = \frac{\sqrt{2}}{(-4)^m}.$$

REFERENCES

1. W.A. Beyer, L.D. Louck, and D. Zeilberger, Math Bite: A generalization of a curiosity that Feynman remembered all his life, this MAGAZINE 69 (1996), 43–44.
2. C.V. Durell and A. Robson, *Advanced Trigonometry*, G. Bell & Sons Ltd; London, 1930.
3. James Gleick, *Genius: The Life and Science of Richard Feynman*, Pantheon Books, New York, NY, 1992.

—GARRY J. TEE
UNIVERSITY OF AUCKLAND
AUCKLAND
NEW ZEALAND