A Combinatorial Proof of the Pythagorean Theorem

EDWARD R. SCHEINERMAN*
The Johns Hopkins University
Baltimore, MD 21218-2689

Can the quintessential theorem of plane geometry—the Pythagorean—be understood and proved as a theorem about finite sets? Yes! Here's how.

There are two steps that take us from the Pythagorean theorem to finite sets. First, we rewrite the Pythagorean theorem as the trigonometric identity $\sin^2 x + \cos^2 x = 1$. Second, we see this identity as an equation involving exponential generating functions. The first step is simple; let us focus on exponential generating functions (see [1] for an excellent discussion of generating functions). Given a sequence of numbers, $a_0, a_1, a_2, \ldots$, its *exponential generating function* is

$$A(x) = \sum_{n=0}^{\infty} a_n \frac{x^n}{n!}.$$

We need only one fact about exponential generating functions, namely, the multiplication rule. Suppose $A(x)$, $B(x)$, and $C(x)$ are the exponential generating functions for sequences $\{a_n\}$, $\{b_n\}$, and $\{c_n\}$, respectively. If $C(x) = A(x)B(x)$, how does c_n relate to the a's and the b's? The answer is the following convolution formula that the reader can easily check:

$$c_n = \sum_{k=0}^{n} \binom{n}{k} a_k b_{n-k}.$$

Thus if we write

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots = \sum_{n=0}^{\infty} s_n \frac{x^n}{n!},$$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots = \sum_{n=0}^{\infty} c_n \frac{x^n}{n!},$$

the Pythagorean theorem ($\sin^2 x + \cos^2 x = 1$) can be written:

$$\sum_{k=0}^{n} \binom{n}{k} s_k s_{n-k} + \sum_{k=0}^{n} \binom{n}{k} c_k c_{n-k} = \begin{cases} 1 & n=0 \\ 0 & n>0 \end{cases}. \tag{1}$$

When $n = 0$ the only nonzero term is $\binom{0}{0} c_0 c_0 = 1$, so let us focus on the case $n > 0$. If n is odd, all the terms $s_k s_{n-k}$ and $c_k c_{n-k}$ are zero, so the statement is trivial. However, when n is even, we check that (1) reduces to

$$\sum_{k=0}^{n} (-1)^k \binom{n}{k} = 0. \tag{2}$$

Now (2) is an immediate consequence of the binomial theorem, but we promised the reader an interpretation about finite sets. Separating positive and negative terms,

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we have

$$\sum_{k \text{ odd}} \binom{n}{k} = \sum_{k \text{ even}} \binom{n}{k} \tag{3}$$

that says: The number of odd cardinality subsets on an n-set equals the number of subsets of even cardinality. To finish, we need a one-to-one correspondence between the odd and even subsets of an n-set. When n is odd, taking complements does the trick. But our job is to prove (2) (or (3)) for n even! This is almost as easy. Let $N = \{1, 2, \dots, n\}$. Then for any $A \subseteq N$, let

$$A' = \begin{cases} A \cup \{1\} & \text{when } 1 \notin A \text{, and} \\ A \setminus \{1\} & \text{when } 1 \in A. \end{cases}$$

Finally, observe that $A \leftrightarrow A'$ gives a one-to-one pairing of the even and the odd subsets of N.

Cute, but is it a proof? Yes. One "only" needs to check that the Taylor series for $\sin x$ and $\cos x$ are correct and bring some real analysis to bear. To this end, we need to know that $\frac{d}{dx}\sin x = \cos x$, $\frac{d}{dx}\cos x = -\sin x$, $\sin 0 = 0$ and $\cos 0 = 1$. The derivatives can be derived from standard formulas such as $\sin(x+y) = \sin x \cos y + \sin y \cos x$ and basic limits such as $\frac{\sin x}{x} \to 1$ as $x \to 0$. These can be verified using geometric arguments that do not require the use of the Pythagorean Theorem.

REFERENCE

1. Herbert S. Wilf, *Generatingfunctionology*, Academic Press, 1990.

Math Bite: Normality of the Commutator Subgroup

The commutator subgroup C of a group G is the smallest subgroup of G containing all elements of the form $aba^{-1}b^{-1}$, where a and b are arbitrary elements of G. To see that C is a normal subgroup of G, let c be a member of C, and let g be any element of G. Note that $gcg^{-1}c^{-1}$ is in C, whence by closure $gcg^{-1}c^{-1}c = gcg^{-1}$ must also belong to C.

—Lawrence Myers
Research Triangle Institute
Research Triangle Park, NC 27709