A New Elementary Proof of Stirling's Formula

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If n is a positive integer, the ratio of the $(k+1)$st term to the kth term in the power series

$$e^n = 1 + n + \frac{n^2}{2!} + \cdots \tag{1}$$

is n/k. Thus, the sequence of terms increases as long as $k < n$ and decreases when $k > n$. The nth and $(n+1)$st terms have the same magnitude, $n^n/n!$, and this is the largest magnitude possible for any term in the series in (1). What is the behavior of the ratio of e^n to the largest term in its power series as $n \to \infty$? This question is answered by Stirling's formula, usually written in the form

$$n! \backsim n^n e^{-n} \sqrt{2\pi n} \quad (n \to \infty). \tag{2}$$

Equation (2) means that

$$\lim_{n\to\infty} \frac{n!}{n^n e^{-n}\sqrt{2\pi n}} = 1.$$

More generally, f is said to be asymptotic to g as $n \to \infty$, written $f(n) \backsim g(n)$ $(n \to \infty)$, if

$$\lim_{n\to\infty} \frac{f(n)}{g(n)} = 1.$$

To answer the question above, we can write (2) as

$$e^n \frac{n!}{n^n} \backsim \sqrt{2\pi n} \qquad (n \to \infty). \tag{3}$$

Stirling's formula appears in many different disciplines, from algorithm analysis to statistical mechanics. Many derivations of it have been given. See, for example, **[1]**, **[2]**, and **[3]**. In this note we present a new elementary proof of (2); on the way we will see that the two largest terms in the power series for e^n asymptotically separate its sum into equal parts as $n \to \infty$. We begin with the power series for e^n in (1) and pare it down to a point where we can conclude (3). The proof requires little beyond first-year calculus. More specifically, it requires the following three results:

$$e^x > 1 + x, \qquad (x \neq 0). \tag{4}$$

This is easily proved by noting that

$$\int_0^x (1 - e^{-t})\,dt > 0, \qquad x \neq 0.$$

If $x > -1$, $x \neq 0$, and $m > 1$ is an integer, then

$$(1 + x)^m > 1 + mx. \tag{5}$$

This result, often called Bernoulli's inequality, is proved easily by induction. The evaluation of the definite integral

$$\int_0^\infty e^{-x^2}\,dx = \frac{\sqrt{\pi}}{2}, \tag{6}$$

not quite within reach of most first-year calculus students, can be performed in the standard way by considering an iterated double integral and changing to polar coordinates (see, for example, [1, p. 128, Exercise 10]).

For $n \geq 1$, the terms in the power series for e^n in (1) are of two types: those of the form $n^{n+k}/(n+k)!$ with $k \geq 0$ and those of the form $n^{n-k-1}/(n-k-1)!$ with $0 \leq k \leq n-1$. When $k=0$ each of these expressions has magnitude $n^n/n!$. To compare their magnitudes to $n^n/n!$ when $k>0$, we let

$$\frac{n^{n-k-1}}{(n-k-1)!} = \frac{n^n}{n!}\alpha_k \quad \frac{n^{n+k}}{(n+k)!} = \frac{n^n}{n!}\beta_k,$$

where

$$\alpha_k = \left(1-\frac{0}{n}\right)\left(1-\frac{1}{n}\right)\cdots\left(1-\frac{k}{n}\right) \tag{7}$$

for $0 \leq k \leq n-1$, and

$$\beta_k = \frac{1}{\left(1+\frac{0}{n}\right)\left(1+\frac{1}{n}\right)\cdots\left(1+\frac{k}{n}\right)}$$

$$= \left(1-\frac{0}{n+0}\right)\left(1-\frac{1}{n+1}\right)\cdots\left(1-\frac{k}{n+k}\right) \tag{8}$$

for $0 \leq k < \infty$. We can then write

$$e^n\frac{n!}{n^n} = \sum_{k=0}^{n-1}\alpha_k + \sum_{k=0}^{\infty}\beta_k. \tag{9}$$

Note that $\alpha_k \leq \beta_k \leq 1$ for $0 \leq k \leq n-1$, and that α_k and β_k are both monotone decreasing functions of k. Using (4) and (8), we can show that

$$\beta_k \leq e^{-\left(\frac{0}{n+0}+\frac{1}{n+1}\cdots+\frac{k}{n+k}\right)}. \tag{10}$$

Now

$$\frac{0}{n+0}+\frac{1}{n+1}\cdots+\frac{k}{n+k} \geq \frac{0+1+\cdots+k}{n+k} \geq \frac{k^2}{2(n+k)}, \tag{11}$$

so that when $k \geq n$ we have $(1/2)k^2/(n+k) \geq k/4$ and (10) and (11) imply

$$\sum_{k=n}^{\infty}\beta_k \leq \sum_{k=n}^{\infty}e^{-k/4} = e^{-n/4}\sum_{k=0}^{\infty}e^{-k/4}. \tag{12}$$

On the other hand, when $k \leq n$, we have $(1/2)k^2/(n+k) \geq k^2/4n$ and (10) and (11) imply

$$\alpha_k \leq \beta_k \leq e^{-k^2/4n}, \; (0 \leq k \leq n-1). \tag{13}$$

This suggests the existence of a k_n between 0 and $n-1$ beyond which α_k and β_k

are asymptotically unimportant. To make this more precise, for $0 < \varepsilon < 1/2$ we define $k_n = \lfloor n^{(1/2)+\varepsilon} \rfloor$, the greatest integer less than or equal to $n^{(1/2)+\varepsilon}$. The integer k_n satisfies the inequalities

$$n^{(1/2)+\varepsilon} - 1 < k_n \leq n^{(1/2)+\varepsilon}.$$

We will see how to select ε to our advantage. For $k_n < k < n-1$, (13) implies that

$$\alpha_k \leq \beta_k < \beta_{k_n+1} < e^{-n^{2\varepsilon}/4}.$$

Summing these inequalities on k then yields

$$\sum_{k=k_n+1}^{n-1} \alpha_k \leq \sum_{k=k_n+1}^{n-1} \beta_k \leq n e^{-n^{2\varepsilon}/4}. \tag{14}$$

A combination of (9), (12), and (14) gives, upon letting $n \to \infty$,

$$e^n \frac{n!}{n^n} \sim \sum_{k=0}^{k_n} \alpha_k + \sum_{k=0}^{k_n} \beta_k \qquad (n \to \infty). \tag{15}$$

The next step is to determine the asymptotic behavior of the right side of (15). From (4) it follows that $1 - x \leq e^{-x} \leq 1/(1+x)$ for $x \geq 0$, and using this in (7) and (8) yields

$$\alpha_k \leq e^{-k(k+1)/2n} \leq \beta_k, \qquad (0 \leq k \leq k_n).$$

This in turn implies

$$\alpha_k \leq e^{-k^2/2n} \leq e^{k_n/2n} \beta_k, \qquad (0 \leq k \leq k_n). \tag{16}$$

Now (7) and (8) also show that

$$\begin{aligned} \frac{\alpha_k}{\beta_k} &= \left[1 - \left(\frac{0}{n}\right)^2\right]\left[1 - \left(\frac{1}{n}\right)^2\right] \cdots \left[1 - \left(\frac{k}{n}\right)^2\right] \\ &\geq \left[1 - \left(\frac{k}{n}\right)^2\right]^k \\ &\geq 1 - \frac{k^3}{n^2}, \end{aligned} \tag{17}$$

for $0 \leq k \leq k_n$, where (5) has been used to obtain the last inequality in (17). Since $\alpha_k \leq \beta_k$ for $0 \leq k \leq k_n$, (17) implies

$$\left[1 - \frac{k_n^3}{n^2}\right] \beta_k \leq \alpha_k \leq \beta_k \tag{18}$$

when $0 \leq k \leq k_n$. The inequalities in (16) and (18) can then be combined and summed from $k = 0$ to $k = k_n$ to give

$$\left[1 - \frac{k_n^3}{n^2}\right] \sum_{k=0}^{k_n} \beta_k \leq \sum_{k=0}^{k_n} \alpha_k \leq \sum_{k=0}^{k_n} e^{-k^2/2n} \leq e^{k_n/2n} \sum_{k=0}^{k_n} \beta_k. \tag{19}$$

Since $k_n/n \leq n^{(-1/2)+\varepsilon}$ and $k_n^3/n^2 \leq n^{(-1/2)+3\varepsilon}$, if we now choose ε so that $0 < \varepsilon < 1/6$, then $k_n/n \to 0$ and $k_n^3/n^2 \to 0$ as $n \to \infty$. We divide each term in (19) by

$\sum_{k=0}^{k_n}\beta_k$ and let $n \to \infty$ to conclude that

$$\sum_{k=0}^{k_n} \alpha_k \sim \sum_{k=0}^{k_n} e^{-k^2/2n} \quad \text{and} \quad \sum_{k=0}^{k_n} \beta_k \sim \sum_{k=0}^{k_n} e^{-k^2/2n}$$

as $n \to \infty$. Consequently (15) becomes

$$e^n \frac{n!}{n^n} \sim 2 \sum_{k=0}^{k_n} e^{-k^2/2n} \qquad (n \to \infty). \tag{20}$$

Together with (14) these equations imply that

$$e^n \frac{n!}{n^n} \sim 2 \sum_{k=0}^{n-1} \alpha_k \qquad (n \to \infty). \tag{21}$$

If we employ the definition of α_k given in (7) and rearrange (21), we obtain

$$1 + \frac{n}{1!} + \frac{n^2}{2!} + \cdots + \frac{n^{n-1}}{(n-1)!} \sim \frac{e^n}{2} \qquad (n \to \infty).$$

Thus we have the remarkable fact that *the two largest terms in the power series for e^n asymptotically separate its sum into equal parts as $n \to \infty$!*

To estimate the sum on the right side of (20) asymptotically, note that the function f expressed by $f(x) = e^{-x^2}$ is positive and monotonically decreasing on $[0, \infty)$. If we set $h = 1/\sqrt{2n}$, it follows that

$$\int_h^{(k_n+1)h} f(x)\,dx \leq \sum_{k=1}^{k_n} hf(kh) \leq \int_0^{k_n h} f(x)\,dx.$$

Since $k_n h \to \infty$ and $h \to 0$ as $n \to \infty$, we let $n \to \infty$ to obtain

$$\lim_{n\to\infty} \frac{1}{\sqrt{2n}} \sum_{k=1}^{k_n} e^{-k^2/2n} = \int_0^\infty e^{-x^2} dx,$$

or, using (6),

$$\sum_{k=1}^{k_n} e^{-k^2/2n} \sim \frac{\sqrt{2\pi n}}{2} \qquad (n \to \infty).$$

It is also clear that

$$\sum_{k=0}^{k_n} e^{-k^2/2n} \sim \frac{\sqrt{2\pi n}}{2} \qquad (n \to \infty), \tag{22}$$

so that by combining (20) and (22) we obtain (3) and hence (2).

REFERENCES

1. Tom M. Apostol, *Calculus*, Vol. 2, Blaisedell Pub. Co., Waltham, MA, 1962.
2. A. J. Coleman, A simple proof of Stirling's formula, *Amer. Math. Monthly* 58 (1951) 334–336.
3. Serge Lang, A *First Course in Calculus*, 4th edition, Addision-Wesley Publishing Co., Reading, MA, 1978.