A New Elementary Proof of Stirling’s Formula

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If \( n \) is a positive integer, the ratio of the \((k + 1)\)st term to the \( k \)th term in the power series

\[
e^n = 1 + n + \frac{n^2}{2!} + \cdots
\]  

is \( n/k \). Thus, the sequence of terms increases as long as \( k < n \) and decreases when \( k > n \). The \( n \)th and \((n + 1)\)st terms have the same magnitude, \( n^n/n! \), and this is the largest magnitude possible for any term in the series in (1). What is the behavior of the ratio of \( e^n \) to the largest term in its power series as \( n \to \infty \)? This question is answered by Stirling’s formula, usually written in the form

\[
n! \sim n^n e^{-n} \sqrt{2\pi n} \quad (n \to \infty).
\]  

Equation (2) means that

\[
\lim_{n \to \infty} \frac{n!}{n^n e^{-n} \sqrt{2\pi n}} = 1.
\]

More generally, \( f \) is said to be asymptotic to \( g \) as \( n \to \infty \), written \( f(n) \sim g(n) \) \( (n \to \infty) \), if

\[
\lim_{n \to \infty} \frac{f(n)}{g(n)} = 1.
\]

To answer the question above, we can write (2) as

\[
e^n \frac{n!}{n^n} \sim \sqrt{2\pi n} \quad (n \to \infty).
\]

Stirling’s formula appears in many different disciplines, from algorithm analysis to statistical mechanics. Many derivations of it have been given. See, for example, [1], [2], and [3]. In this note we present a new elementary proof of (2); on the way we will see that the two largest terms in the power series for \( e^n \) asymptotically separate its sum into equal parts as \( n \to \infty \). We begin with the power series for \( e^n \) in (1) and pare it down to a point where we can conclude (3). The proof requires little beyond first-year calculus. More specifically, it requires the following three results:

\[
e^x > 1 + x, \quad (x \neq 0).
\]

This is easily proved by noting that

\[
\int_0^x (1 - e^{-t}) \, dt > 0, \quad x \neq 0.
\]

If \( x > -1 \), \( x \neq 0 \), and \( m > 1 \) is an integer, then

\[
(1 + x)^m > 1 + mx.
\]
This result, often called Bernoulli’s inequality, is proved easily by induction. The evaluation of the definite integral
\[ \int_0^\infty e^{-x^2} \, dx = \frac{\sqrt{\pi}}{2}, \] (6)
not quite within reach of most first-year calculus students, can be performed in the standard way by considering an iterated double integral and changing to polar coordinates (see, for example, [1, p. 128, Exercise 10]).

For \( n \geq 1 \), the terms in the power series for \( e^n \) in (1) are of two types: those of the form \( n^{n+k}/(n+k)! \) with \( k \geq 0 \) and those of the form \( n^{n-k-1}/(n-k-1)! \) with \( 0 \leq k \leq n-1 \). When \( k = 0 \) each of these expressions has magnitude \( n^n/n! \). To compare their magnitudes to \( n^n/n! \) when \( k > 0 \), we let
\[ \frac{n^{n-k-1}}{(n-k-1)!} = \frac{n^n}{n!} \alpha_k \quad \frac{n^{n+k}}{(n+k)!} = \frac{n^n}{n!} \beta_k, \]
where
\[ \alpha_k = \left(1 - \frac{0}{n}\right)\left(1 - \frac{1}{n}\right) \cdots \left(1 - \frac{k}{n}\right) \] (7)
for \( 0 \leq k \leq n-1 \), and
\[ \beta_k = \frac{1}{\left(1 + \frac{0}{n}\right)\left(1 + \frac{1}{n}\right) \cdots \left(1 + \frac{k}{n}\right)} = \frac{1}{\left(1 - \frac{0}{n+0}\right)\left(1 - \frac{1}{n+1}\right) \cdots \left(1 - \frac{k}{n+k}\right)} \]
(8)
for \( 0 \leq k < \infty \). We can then write
\[ e^n \frac{n!}{n^n} = \sum_{k=0}^{n-1} \alpha_k + \sum_{k=0}^\infty \beta_k. \] (9)
Note that \( \alpha_k \leq \beta_k \leq 1 \) for \( 0 \leq k \leq n-1 \), and that \( \alpha_k \) and \( \beta_k \) are both monotone decreasing functions of \( k \). Using (4) and (8), we can show that
\[ \beta_k \leq e^{-\left(\frac{0}{n+0} + \frac{1}{n+1} \cdots + \frac{k}{n+k}\right)} \]
(10)
Now
\[ \frac{0}{n+0} + \frac{1}{n+1} \cdots + \frac{k}{n+k} \geq 0 + 1 + \cdots + k \geq \frac{k^2}{2(n+k)}, \]
(11)
so that when \( k \geq n \) we have \((1/2)k^2/(n+k) \geq k/4\) and (10) and (11) imply
\[ \sum_{k=n}^\infty \beta_k \leq \sum_{k=n}^\infty e^{-k/4} = e^{-n/4} \sum_{k=0}^\infty e^{-k/4}. \] (12)
On the other hand, when \( k \leq n \), we have \((1/2)k^2/(n+k) \geq k^2/4n\) and (10) and (11) imply
\[ \alpha_k \leq \beta_k \leq e^{-k^2/4n}, \quad (0 \leq k \leq n-1). \] (13)
This suggests the existence of a \( k_n \) between 0 and \( n-1 \) beyond which \( \alpha_k \) and \( \beta_k \)
are asymptotically unimportant. To make this more precise, for $0 < \varepsilon < 1/2$ we define $k_n = \lfloor n^{(1/2) + \varepsilon} \rfloor$, the greatest integer less than or equal to $n^{(1/2) + \varepsilon}$. The integer $k_n$ satisfies the inequalities

$$n^{(1/2) + \varepsilon} - 1 < k_n \leq n^{(1/2) + \varepsilon}.$$  

We will see how to select $\varepsilon$ to our advantage. For $k_n < k < n - 1$, (13) implies that

$$\alpha_k \leq \beta_k < \beta_{k_n + 1} < e^{-n^{2\varepsilon}/4}.$$  

Summing these inequalities on $k$ then yields

$$\sum_{k=k_n+1}^{n-1} \alpha_k \leq \sum_{k=k_n+1}^{n-1} \beta_k \leq ne^{-n^{2\varepsilon}/4}. \tag{14}$$  

A combination of (9), (12), and (14) gives, upon letting $n \to \infty$,

$$e^n n^1/n^n \sim \sum_{k=0}^{k_n} \alpha_k + \sum_{k=0}^{k_n} \beta_k \quad (n \to \infty). \tag{15}$$  

The next step is to determine the asymptotic behavior of the right side of (15). From (4) it follows that $1 - x \leq e^{-x} \leq 1/(1 + x)$ for $x \geq 0$, and using this in (7) and (8) yields

$$\alpha_k \leq e^{-k(k+1)/2n} \leq \beta_k, \quad (0 \leq k \leq k_n).$$  

This in turn implies

$$\alpha_k \leq e^{-k^2/2n} \leq e^{k_n^2/2n} \beta_k, \quad (0 \leq k \leq k_n). \tag{16}$$  

Now (7) and (8) also show that

$$\frac{\alpha_k}{\beta_k} = \left[ 1 - \left( \frac{0}{n} \right) ^2 \right] \left[ 1 - \left( \frac{1}{n} \right) ^2 \right] \cdots \left[ 1 - \left( \frac{k}{n} \right) ^2 \right]$$

$$\geq \left[ 1 - \left( \frac{k}{n} \right) ^2 \right] ^k$$

$$\geq 1 - \frac{k^3}{n^2}, \tag{17}$$  

for $0 \leq k \leq k_n$, where (5) has been used to obtain the last inequality in (17). Since $\alpha_k \leq \beta_k$ for $0 \leq k \leq k_n$, (17) implies

$$\left[ 1 - \frac{k^3}{n^2} \right] \beta_k \leq \alpha_k \leq \beta_k \tag{18}$$  

when $0 \leq k \leq k_n$. The inequalities in (16) and (18) can then be combined and summed from $k = 0$ to $k = k_n$ to give

$$\left[ 1 - \frac{k_n^3}{n^2} \right] \sum_{k=0}^{k_n} \beta_k \leq \sum_{k=0}^{k_n} \alpha_k \leq \sum_{k=0}^{k_n} e^{-k^2/2n} \leq e^{k_n^2/2n} \sum_{k=0}^{k_n} \beta_k. \tag{19}$$  

Since $k_n/n \leq n^{(-1/2) + \varepsilon}$ and $k_n^3/n^2 \leq n^{(-1/2) + 3\varepsilon}$, if we now choose $\varepsilon$ so that $0 < \varepsilon < 1/6$, then $k_n/n \to 0$ and $k_n^3/n^2 \to 0$ as $n \to \infty$. We divide each term in (19) by
\[ \sum_{k=0}^{k_n} \alpha_k \sim \sum_{k=0}^{k_n} e^{-k^2/2n} \quad \text{and} \quad \sum_{k=0}^{k_n} \beta_k \sim \sum_{k=0}^{k_n} e^{-k^2/2n} \]
as \( n \to \infty \). Consequently (15) becomes
\[ e^n \frac{n!}{n^n} \sim 2 \sum_{k=0}^{k_n} e^{-k^2/2n} \quad (n \to \infty). \] (20)
Together with (14) these equations imply that
\[ e^n \frac{n!}{n^n} \sim 2 \sum_{k=0}^{n-1} \alpha_k \quad (n \to \infty). \] (21)
If we employ the definition of \( \alpha_k \) given in (7) and rearrange (21), we obtain
\[ 1 + \frac{n}{1!} + \frac{n^2}{2!} + \cdots + \frac{n^{n-1}}{(n-1)!} \sim \frac{e^n}{2} \quad (n \to \infty). \]
Thus we have the remarkable fact that the two largest terms in the power series for \( e^n \) asymptotically separate its sum into equal parts as \( n \to \infty \).
To estimate the sum on the right side of (20) asymptotically, note that the function \( f \) expressed by \( f(x) = e^{-x^2} \) is positive and monotonically decreasing on \([0, \infty)\). If we set \( h = 1/\sqrt{2n} \), it follows that
\[ \int_h^{(k_n+1)h} f(x) \, dx \leq \sum_{k=1}^{k_n} hf(kh) \leq \int_0^{k_n h} f(x) \, dx. \]
Since \( k_n h \to \infty \) and \( h \to 0 \) as \( n \to \infty \), we let \( n \to \infty \) to obtain
\[ \lim_{n \to \infty} \frac{1}{\sqrt{2n}} \sum_{k=1}^{k_n} e^{-k^2/2n} = \int_0^{\infty} e^{-x^2/2} \, dx, \]
or, using (6),
\[ \sum_{k=1}^{k_n} e^{-k^2/2n} \sim \frac{\sqrt{2 \pi n}}{2} \quad (n \to \infty). \]
It is also clear that
\[ \sum_{k=0}^{k_n} e^{-k^2/2n} \sim \frac{\sqrt{2 \pi n}}{2} \quad (n \to \infty), \] (22)
so that by combining (20) and (22) we obtain (3) and hence (2).

REFERENCES