Exploring Complex-Base Logarithms

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Students of complex analysis soon discover that the natural logarithm is a multivalued function with an infinite number of branches, reflecting the multiple representation of any complex number. If $z = x + iy \equiv (x, y)$, then in polar form:

$$z = re^{i[\theta + 2\pi n]}$$

with $r = +\sqrt{x^2 + y^2}$, $0 \leqslant \theta \leqslant \tan^{-1}(y/x) < 2\pi$, and n any integer. Hence

$$w = \ln(z) = \ln(r) + i(\theta + 2\pi n).$$

Interesting patterns arise when multivalued logarithms are generalized to arbitrary complex bases. To complex base z_1 the logarithm,

$$w = \log_{z_1}(z_2), \tag{1}$$

is defined by

$$z_2 = z_1^w. \tag{2}$$

Consider now that both z_1 and z_2 have multiple representations:

$$z_1 = \rho e^{i[\phi + 2\pi m]}$$
$$z_2 = re^{i[\theta + 2\pi n]}.$$

Taking the natural logarithm of (2), (1) is equivalent to:

$$w = \frac{\ln(r) + i(\theta + 2\pi n)}{\ln(\rho) + i(\phi + 2\pi m)}. \tag{3}$$

As m and n independently run through all integers, (3) defines an infinite number of points in the complex plane that serve as representations of the logarithm of z_2 to base z_1. What pattern do they describe?

Fix an integer m, and let $C_m = \phi + 2\pi m$. Then, as n varies, (3) gives for $w = x + iy$:

$$\begin{aligned} x &= [\ln(r) \cdot \ln(\rho) + C_m \cdot (\theta + 2\pi n)] / [\ln^2(\rho) + C_m^2] \\ y &= [\ln(\rho) \cdot (\theta + 2\pi n) - \ln(r) \cdot C_m] / [\ln^2(\rho) + C_m^2]. \end{aligned} \tag{4}$$

Replace $(\theta + 2\pi n)$ by a continuous variable s that, when eliminated from (4), gives

$$y = x \cdot [\ln(\rho)/C_m] - \ln(r)/C_m. \tag{5}$$

That is, for fixed m, points (in the Cartesian plane) corresponding to the base z_1 logarithm of z_2 fall on a straight line of slope $[\ln(\rho)/C_m]$ and intercept $[-\ln(r)/C_m]$; they are points for which $(s - \theta)/2\pi$ is an integer. Note that all lines defined by (5) pass through the point $A = (\ln(r)/\ln(\rho), 0)$, independently of m.

Now fix n in (3) and let $D_n = (\theta + 2\pi n)$; replace $(\phi + 2\pi m)$ by a continuous variable s to get a parametric expression for the locus of points on which logarithm representations fall as m varies:

$$\begin{aligned} x &= [\ln(r)\cdot\ln(\rho) + D_n\cdot s]/[\ln^2(\rho)+s^2] \\ y &= [\ln(\rho)\cdot D_n - \ln(r)\cdot s]/[\ln^2(\rho)+s^2]. \end{aligned} \tag{6}$$

Eliminating s from (6) is algebraically more unwieldy than for the fixed m case. Anyone who plots several examples, however, will no doubt speculate that (6) is a circle in the Cartesian plane. Substitute (6) into the general equation for a circle with center (u, v) and radius R:

$$(x-u)^2 + (y-v)^2 = R^2,$$

and solve for u, v, and R by evaluating at three convenient values of s (for example, $s = 0$, $\ln(\rho)D_n/\ln(r)$, and ∞), to show that (6) is equivalent to

$$\left[x - \frac{\ln(r)}{2\ln(\rho)}\right]^2 + \left[y - \frac{D_n}{2\ln(\rho)}\right]^2 = \frac{\ln^2(r)+D_n^2}{4\ln^2(\rho)}. \tag{7}$$

As n varies through integer values, (7) defines a family of circles, each with center on the vertical line $x = \ln(r)/[2\ln(\rho)]$, and passing through the point A. The fact that each circle (7) intersects each line (5) at A, guarantees another intersection point, which is the logarithm representation for the corresponding (m, n) pair.

FIGURE 1 shows the geometry of intersecting lines and circles for $z_1 = (3, 2)$, $z_2 = (2, -1)$, and $|m|, |n| \leqslant 2$; curves are labeled with their m, n values. The point of common intersection is $A = (.627, 0)$; all other intersection points of individual lines and circles are representations of w. FIGURE 2 is the same for $z_1 = (-1, 3)$, $z_2 = (.1, .1)$, with $A = (-1.699, 0)$.

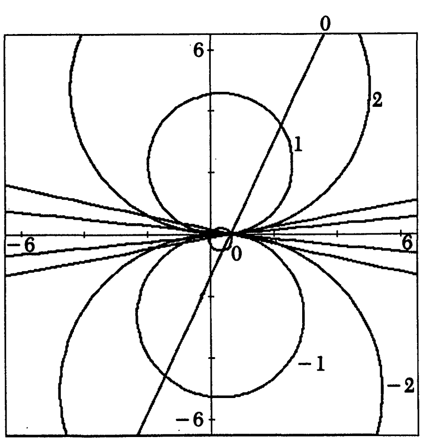

FIGURE 1

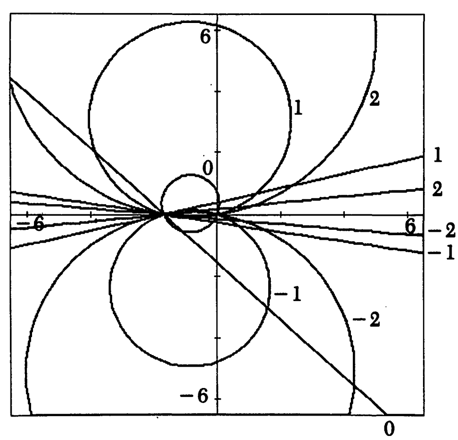

FIGURE 2

For fixed m, (4) shows that both $|x|$ and $|y| \to \infty$ as $|n| \to \infty$, so there are logarithm points arbitrarily far from the origin, along each line (5). For fixed n, the circle (7) intersects every line (5); as $|m| \to \infty$ their slopes approach zero, so that the intersection points approach the x (real) axis. FIGURES 3 through 6 display this concentration of logarithm points about the x-axis, for the case $z_1 = (3, 2)$, $z_2 = (2, 1)$. For FIGURE 3, the plot limits are $0 \leqslant x \leqslant 2$; $-.1 \leqslant y \leqslant .1$ (note distortion), and $|m|, |n| \leqslant 25$. Each successive figure is a blowup of the small boxed region of the preceding figure. Truncation values for FIGURES 4 through 6 are $|m|$, $|n| \leqslant 50$, 200, and 1000, respectively. Truncation at finite m, n leads to the point-free central band of each figure.

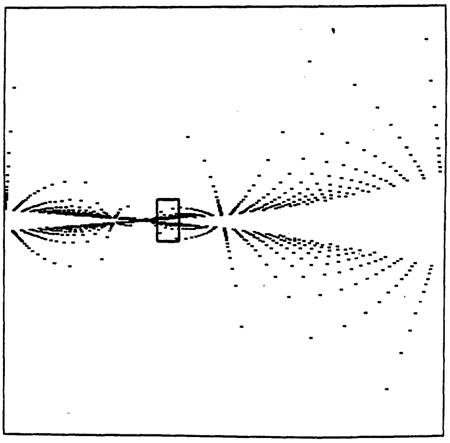

FIGURE 3

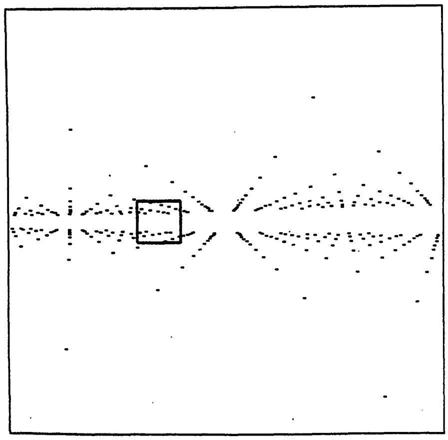

FIGURE 4

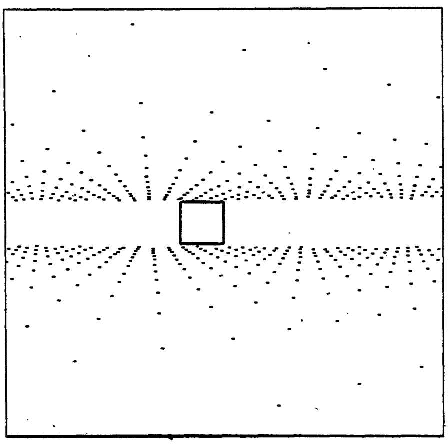

FIGURE 5

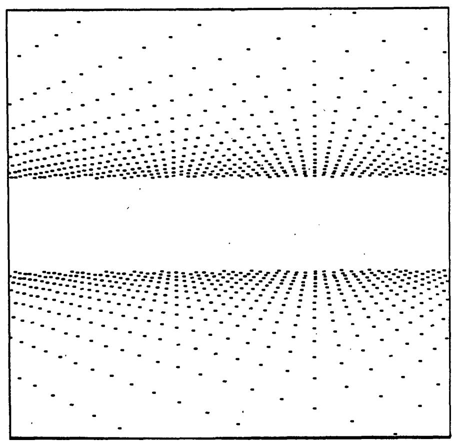

FIGURE 6

This elegant pattern of fractal-like ray structures, upon which complex-base logarithms fall, is a particular example of patterns assumed by multiple representations of arbitrary functions of k complex numbers:

$$w = f(z_1, z_2, \dots, z_k).$$

For example, Gleason [1; p. 324] explicitly comments on the infinite number of representations of

$$w = \ln(z_1 z_2).$$

In this case, however, the pattern is far less interesting. Expressing z_1, z_2 as above,

$$w = \ln(r\rho) + i[\theta + \phi + 2\pi(n+m)]$$

and all representations fall at equal spacing on the single line $x = \ln(r\rho)$.

REFERENCE

1. Andrew M. Gleason, *Fundamentals of Abstract Analysis*, Jones and Bartlett Publishers, Boston, 1991.