Exploring Complex-Base Logarithms

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Students of complex analysis soon discover that the natural logarithm is a multivalued function with an infinite number of branches, reflecting the multiple representation of any complex number. If \( z = x + iy = (x, y) \), then in polar form:

\[
z = re^{i(\theta + 2\pi n)}
\]

with \( r = +\sqrt{x^2 + y^2} \), \( 0 \leq \theta \leq \tan^{-1}(y/x) < 2\pi \), and \( n \) any integer. Hence

\[
w = \ln(z) = \ln(r) + i(\theta + 2\pi n).
\]

Interesting patterns arise when multivalued logarithms are generalized to arbitrary complex bases. To complex base \( z_1 \) the logarithm,

\[
w = \log_{z_1}(z_2),
\]

is defined by

\[
z_2 = z_1^w.
\]

Consider now that both \( z_1 \) and \( z_2 \) have multiple representations:

\[
z_1 = \rho e^{i(\phi + 2\pi m)}
\]

\[
z_2 = re^{i(\theta + 2\pi n)}.
\]

Taking the natural logarithm of (2), (1) is equivalent to:

\[
w = \frac{\ln(r) + i(\theta + 2\pi n)}{\ln(\rho) + i(\phi + 2\pi m)}.
\]

As \( m \) and \( n \) independently run through all integers, (3) defines an infinite number of points in the complex plane that serve as representations of the logarithm of \( z_2 \) to base \( z_1 \). What pattern do they describe?

Fix an integer \( m \), and let \( C_m = \phi + 2\pi m \). Then, as \( n \) varies, (3) gives for \( w = x + iy \):

\[
x = \left[ \ln(r) \cdot \ln(\rho) + C_m \cdot (\theta + 2\pi n) \right] / \left[ \ln^2(\rho) + C_m^2 \right]
\]

\[
y = \left[ \ln(\rho) \cdot (\theta + 2\pi n) - \ln(r) \cdot C_m \right] / \left[ \ln^2(\rho) + C_m^2 \right].
\]

Replace \( (\theta + 2\pi n) \) by a continuous variable \( s \) that, when eliminated from (4), gives

\[
y = x \cdot \left[ \ln(\rho) / C_m \right] - \ln(r) / C_m.
\]

That is, for fixed \( m \), points (in the Cartesian plane) corresponding to the base \( z_1 \) logarithm of \( z_2 \) fall on a straight line of slope \( [\ln(\rho) / C_m] \) and intercept \( [-\ln(r) / C_m] \); they are points for which \( (s - \theta) / 2\pi \) is an integer. Note that all lines defined by (5) pass through the point \( A = (\ln(r) / \ln(\rho), 0) \), independently of \( m \).

Now fix \( n \) in (3) and let \( D_n = (\theta + 2\pi n) \); replace \((\phi + 2\pi m)\) by a continuous variable \( s \) to get a parametric expression for the locus of points on which logarithm
representations fall as \( m \) varies:

\[
\begin{align*}
  x &= \frac{[\ln(r) \cdot \ln(\rho) + D_n \cdot s]}{[\ln^2(\rho) + s^2]} \quad [\ln^2(\rho) + s^2] \\
  y &= \frac{[\ln(\rho) \cdot D_n - \ln(r) \cdot s]}{[\ln^2(\rho) + s^2]}.
\end{align*}
\]  

(6)

Eliminating \( s \) from (6) is algebraically more unwieldy than for the fixed \( m \) case. Anyone who plots several examples, however, will no doubt speculate that (6) is a circle in the Cartesian plane. Substitute (6) into the general equation for a circle with center \((u, v)\) and radius \( R\):

\[
(x - u)^2 + (y - v)^2 = R^2,
\]

and solve for \( u, v \), and \( R \) by evaluating at three convenient values of \( s \) (for example, \( s = 0, \ln(\rho) D_n/\ln(r), \) and \( \infty \)), to show that (6) is equivalent to

\[
\left[ x - \frac{\ln(r)}{2\ln(\rho)} \right]^2 + \left[ y - \frac{D_n}{2\ln(\rho)} \right]^2 = \frac{\ln^2(r) + D_n^2}{4\ln^2(\rho)}.
\]

(7)

As \( n \) varies through integer values, (7) defines a family of circles, each with center on the vertical line \( x = \ln(r)/[2\ln(\rho)] \), and passing through the point \( A \). The fact that each circle (7) intersects each line (5) at \( A \), guarantees another intersection point, which is the logarithm representation for the corresponding \((m, n)\) pair.

Figure 1 shows the geometry of intersecting lines and circles for \( z_1 = (3, 2), z_2 = (2, -1) \), and \( |m|, |n| \leq 2 \); curves are labeled with their \( m, n \) values. The point of common intersection is \( A = (0.627, 0) \); all other intersection points of individual lines and circles are representations of \( w \). Figure 2 is the same for \( z_1 = (-1, 3), z_2 = (1, 1) \), with \( A = (-1.699, 0) \).

![Figure 1](image1.png)

![Figure 2](image2.png)

For fixed \( m \), (4) shows that both \( |x| \) and \( |y| \to \infty \) as \( |n| \to \infty \), so there are logarithm points arbitrarily far from the origin, along each line (5). For fixed \( n \), the circle (7) intersects every line (5); as \( |m| \to \infty \) their slopes approach zero, so that the intersection points approach the \( x \) (real) axis. Figures 3 through 6 display this concentration of logarithm points about the \( x \)-axis, for the case \( z_1 = (3, 2), z_2 = (2, 1) \). For Figure 3, the plot limits are \( 0 \leq x \leq 2; -1 \leq y \leq 1 \) (note distortion), and \( |m|, |n| \leq 25 \). Each successive figure is a blowup of the small boxed region of the preceding figure. Truncation values for Figures 4 through 6 are \( |m|, |n| \leq 50, 200, \) and 1000, respectively. Truncation at finite \( m, n \) leads to the point-free central band of each figure.
This elegant pattern of fractal-like ray structures, upon which complex-base logarithms fall, is a particular example of patterns assumed by multiple representations of arbitrary functions of $k$ complex numbers:

$$w = f(z_1, z_2, \ldots, z_k).$$

For example, Gleason [1; p. 324] explicitly comments on the infinite number of representations of

$$w = \ln(z_1 z_2).$$

In this case, however, the pattern is far less interesting. Expressing $z_1, z_2$ as above,

$$w = \ln(r \rho) + i[\theta + \phi + 2\pi(n + m)]$$

and all representations fall at equal spacing on the single line $x = \ln(r \rho)$.

REFERENCE