Convergence of Complex Continued Fractions

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I. Introduction Our major goal in this paper is to use a variety of techniques available to the advanced undergraduate in the study of the convergence and divergence of a particular complex continued fraction. The material was actually presented to an independent study group and touches upon several areas:

a) analysis—using the derivative to determine the convergence of an iterative sequence;
b) algebra—the use of finitely generated groups in the study of the divergence of periodic sequences;
c) number theory—how counting functions can determine the size of certain groups;
d) topology—changing the setting of a problem to one that easily lends itself to the study of our problem; and
e) complex variables—using properties of Moebius transformations when examining the iterates of a function.

The problem before us is to determine those complex numbers c, for which the continued fraction

$$\cfrac{1}{1+\cfrac{c}{1+\cfrac{c}{\ddots}}} \tag{1}$$

converges. By considering iterates of the function

$$f_c(z) = \frac{1}{1+cz} \tag{2}$$

evaluated at zero; that is, by considering the sequence $f_c(0), f_c(f_c(0)) = f_c^2(0), f_c(f_c(f_c(0))) = f_c^3(0), \ldots, f_c^n(0), \ldots,$ it will be shown that the complex fraction (1) converges for all complex numbers $\mathbb{C}$ except those on the real line less than $-1/4$. We will do this in two ways; in Section II we will use techniques that illustrate (a) and (e), and in Section III we will use techniques illustrating (b), (c), and (d).

The earliest record of real continued fractions is contained in the works of Bombelli and Cataldi [9, p. 1] and dates back to the latter half of the 1500s. However, it wasn't until the 1700s that a systematic treatment of continued fractions was presented by Euler [7] in his book *Introduction to Analysis of the Infinite*. The first known result for complex continued fractions, which includes the type we are examining, dates back to Worpitzky [9, p. 10] in 1865. His theorem dealt with circular regions of convergence. A more generalized statement of the problem, its different cases, and more complete solutions were formalized around the early 1900s by Van Vleck [9, p. 10], Pringsheim [9, p. 46], and others [9, p. 46]. We choose not to deal with the problem in its complete generality for a number of reasons. First, it is not how mathematics is usually done. Generalizations follow from the special cases, and so it would be misleading to the student to do otherwise. Second, generalizations often cloud the picture. The essential ideas in the proof are often buried somewhere in the generalized argument. Third, it makes exploring other techniques of proof and applying them to our setting more difficult.

In addition to the convergence problem, we will also consider what happens to those c-values in $(-\infty, -1/4)$, for which the iterates of $f_c(0)$ diverge. In this interval we define for each $k = 2, 3, \ldots,$ a set D_k, which contains those values of c for which the sequence $\{f_c^n(0)\}$ has k convergent subsequences $\{f_c^{nk+j}(0)\}$, $j = 1, 2, \ldots, k$, each with a distinct limit. It will be shown that the D_k's have a finite number of elements and that their union is dense on $(-\infty, -1/4)$. Major work in the divergence of iterates was done by Julia and Fatou in the 1920s, and with the use of high-power computers continues to be an open and exciting area of study in dynamical systems [3], [5], [6].

II. The derivative and iterates of $f_c(z)$ The techniques we use in the proofs of this section are motivated by the paper of Baker and Rippon [2] in which the convergence of $a, a^a, a^{a^a}, \ldots, a \in \mathbb{C}$, is considered.

THEOREM 1. *If $f_c^n(0)$ converges as $n \to \infty$, then*

$$c \in D = \{t^2 + t\colon \mathrm{Re}(t) > -1/2\} \cup \{-1/4\}.$$

Note that $D = \mathbb{C} \setminus (-\infty, -1/4)$.

Proof. If $c = 0$, then $f_c^n(0) = 1$ for all n. Hence, $\{f_c^n(0)\}$ converges and $c \in D$. Suppose $c \neq 0$ and let $w = \lim_{n \to \infty} f_c^n(0)$. Then w is a fixed point of $f_c(z)$, so $w = 1/(1 + cw)$. Letting $t = cw$ we have $w = 1/(1 + t)$ and $c = t^2 + t$. Because $f_c(z)$ is one-to-one and $w \neq 0$ is a fixed point, $f_c^n(0) \neq w$ for every n, and

$$\lim_{n\to\infty}\left(\frac{f_c^{n+1}(0) - w}{f_c^n(0) - w}\right) = \lim_{n\to\infty}\left(\frac{f_c(f_c^n(0)) - f_c(w)}{f_c^n(0) - w}\right) = f_c'(w) = \frac{-t}{t+1}.$$

Since $\{f_c^n(0)\}$ converges, it must be that $|-t/(t+1)| \leq 1$. Otherwise, one can show that there exists a $\lambda > 1$ such that for n sufficiently large, $|f_c^{n+1}(0) - w| > \lambda |f_c^n(0) - w|$, and this implies that the sequence does not converge to w. Therefore, $|-t/(t+1)| \leq 1$, and this is equivalent to $\mathrm{Re}(t) \geq -1/2$.

We now show that the assumption $\{f_c^n(0)\}$ converges implies that either $\mathrm{Re}(t) > -1/2$ or $t = -1/2$. Suppose to the contrary that $\mathrm{Re}(t) = -1/2$ and $\mathrm{Im}(t) \neq 0$. Then c is real, and $c < -1/4$. Since w is a fixed point of $f_c(z)$, $f_c(w) = w = 1/(1 + cw)$ and it follows that

$$cw^2 + w - 1 = 0, \tag{3}$$

which in turn implies

$$w = \frac{-1 \pm \sqrt{1 + 4c}}{2c}. \tag{4}$$

Thus, w has a nonzero imaginary part. But because c is real-valued, $\{f_c^n(0)\}$ must be contained in $\mathbb{R}$ and so must its limit point, w. This is a contradiction. Hence, our claim is established and the theorem follows.

In our proof the assumption of convergence of the iterates placed a bound less than one on the magnitude of the derivative. Can we argue the result in the other direction? The answer is yes and is the central idea in the proof of the converse of

Theorem 1. We also take advantage of the following properties of Moebius transformations:

i) The composition of Moebius transformations is a Moebius transformation [**1**, p. 77].
ii) A Moebius transformation has at most two fixed points, unless it is the identity function [**1**, p. 78].

THEOREM 2. *If* $c \in D = \{t^2 + t\colon \operatorname{Re}(t) > -1/2\} \cup \{-1/4\}$, *then the sequence* $\{f_c^n(0)\}$, $n = 1, 2, 3, \ldots$ *converges.*

Proof. If $c = 0$, then $f_c^n(0) = 1$ for all n, so the sequence converges. If $c = -1/4$, then

$$f_c(z) = \frac{1}{1 - \frac{1}{4}z}$$

has one fixed point at $z = 2$. Consider the real-valued function

$$F(x) = \frac{1}{1 - \frac{1}{4}x}.$$

Clearly, $F^n(0) = f_c^n(0)$ for all n. Since $F(x)$ is increasing on $(-\infty, 4)$ we have that $F^n(0) < F^n(2) = 2$ for every n. In addition, $F(0) = 1 < 4/3 = F^2(0)$ and so $F^n(0) < F^{n+1}(0)$ for all n. Hence, the sequence $\{F^n(0)\}$ converges. Suppose λ is the limit of this bounded increasing sequence. This gives

$$\lambda = \lim_{k\to\infty} f_c^{k+1}(0) = \lim_{k\to\infty} f_c\big(f_c^k(0)\big) = f_c\Big(\lim_{k\to\infty} f_c^k(0)\Big) = f_c(\lambda).$$

Since there is only one fixed point, we conclude $\lambda = 2$. Therefore, the sequence converges to the point $\lambda = 2$.

Suppose $c \neq 0$ and $c \neq -1/4$. There is exactly one t such that $c = t^2 + t$ and $\operatorname{Re}(t) > -1/2$; hence $f_c(z)$ has two distinct fixed points at $1/(1+t)$ and $-(1/t)$. Let $\Omega = \{z\colon \exists \delta > 0 \ni f_c^n$ converges uniformly to $1/(1+t)$, a constant function, on $N_\delta(z)\}$, where $N_\delta(z)$ denotes an open disk about z of radius δ. Observe that Ω is open. Since $w = 1/(1+t)$ is a fixed point and $|f_c'(w)| = \eta < 1$, one can find a $\delta > 0$ so that when $z \in N_\delta(w)$, $|f_c^n(z) - w| = |f_c^n(z) - f_c^n(w)| < \eta^n|z - w| < \eta^n\delta$. It follows that f_c^n converges uniformly to w in $N_\delta(z)$. Thus, $w = 1/(1+t) \in \Omega$ and obviously $-(1/t) \notin \Omega$.

Let $g_c(z) = (1 - z)/cz$ be the inverse of $f_c(z)$. Since the singularity of $f_c(z)$ is $-(1/c)$, the singularity of $f_c^{n+1}(z)$, which is also a Moebius transformation, is $g_c^n(-(1/c))$. We now show that

$$\lim_{n\to\infty} g_c^n\left(-\frac{1}{c}\right) = -\frac{1}{t}.$$

(We will need this result later.) Let $c_n = g_c^n(-(1/c))$ for each n. Using (i) we can express $f_c^n(z)$ in the form

$$f_c^n(z) = A_n + \frac{B_n}{z - c_{n-1}}, \tag{5}$$

and since $1/(1+t)$ and $-1/t$ are fixed points of f_c^n, we have

$$A_n = -\frac{1}{t} + \frac{1}{t+1} - c_{n-1} \quad\text{and}\quad B_n = \left(\frac{1}{t} + c_{n-1}\right)\left(\frac{1}{t+1} - c_{n-1}\right).$$

In addition,

$$|f_c^{n'}(1/(1+t))| = |f_c'(1/(1+t))|^n = |(-t/(1+t))|^n \to 0 \text{ as } n \to \infty.$$

If we use (5) in this last expression, then

$$\left| \frac{-B_n}{\left(\frac{1}{1+t} - c_{n-1}\right)^2} \right| = \frac{\left|\frac{1}{t} + c_{n-1}\right|}{\left|\frac{1}{1+t} - c_{n-1}\right|} \to 0 \text{ as } n \to \infty.$$

This shows c_n converges to $-1/t$ as $n \to \infty$.

We use this last result to argue that $\Omega = \mathbb{C} \setminus \{-(1/t)\}$. Let $\varepsilon > 0$ be given and consider $K_\varepsilon \equiv \mathbb{C} \setminus \overline{N}_\varepsilon(-(1/t))$, where the bar denotes closure. Since the sequence $\{c_n\}$ converges to $-(1/t)$, $\{A_n\}$ and $\{B_n\}$ converge respectively to $1/(1+t)$ and 0 as $n \to \infty$. So by choosing $N > 0$ such that $|c_n - (-(1/t))| < \varepsilon/2$ for $n > N$, it follows that for any $z \in K_\varepsilon$, $|z - c_n| > \varepsilon/2$ for $n > N$. From (5) and the above comments, we conclude

$$\left| f_c^n(z) - \frac{1}{1+t} \right| \text{ converges uniformly to 0 on } K_\varepsilon,$$

and so $K_\varepsilon \subseteq \Omega$. Upon letting $\varepsilon \to 0$, we see that $\Omega = \mathbb{C} \setminus \{-(1/t)\}$; consequently $0 \in \Omega$ and $\{f_c^n(0)\}$ converges.

The combination of Theorems 1 and 2 insures $\{f_c^n(0)\}$ converges if, and only if, $c \in D = \{t^2 + t: \mathrm{Re}(t) > -1/2\} \cup \{-1/4\}$.

We now turn our attention to the sets D_k, $k = 2, 3, \ldots$. Recall, D_k is the set of real numbers c less than $-1/4$ for which the k subsequences, $\{f_c^{nk+j}(0)\}$, $j = 1, 2, \ldots, k$ as $n \to \infty$, converge to distinct limits. We have the following result.

THEOREM 3. *For each $k \geq 2$, D_k is finite.*

Proof. We first show that D_2 is empty. Suppose not, and let $c \in D_2$. Then $\{f_c^{2n}(0)\}$ converges to a point w, which is a fixed point of $f_c^2(z)$. Since $c \in \mathbb{R}$, $\{f_c^{2n}(0)\} \subseteq \mathbb{R}$, and therefore, w must also be real-valued. But, upon inspection, $w = f_c^2(w)$ implies Equations (3) and (4). Thus, w has a nonzero imaginary part, which is a contradiction. Therefore, D_2 is empty.

Since $f_c^k(z)$ is a Moebius transformation, it has at most two fixed points. Now for each c in D_k, $k > 2$, $\{f_c^n(0)\}$ has k convergent subsequences $\{f_c^{nk+j}(0)\}$, $j = 1, 2, \ldots, k$, each having a distinct limit. These k limits are fixed points of $f_c^k(z)$. By (ii), we must have $f_c^k(z) \equiv z$. In the case that $c \in D_3$,

$$f_c^3(z) = \cfrac{1}{1 + \cfrac{c}{1 + \cfrac{c}{1 + cz}}} = z, \quad \forall z.$$

This simplifies to $(c+1)(cz^2 + z - 1) = 0$ for all z. Since the second factor cannot be 0 for all z, it follows that $c = -1$ and so $D_3 = \{-1\}$. In general, if $c \in D_k$, then $f_c^k(z) = z$ for all z. Rewriting this as $f_c(f_c^{k-1}(z)) = z$, using (5) on $f_c^{k-1}(z)$, and the expressions for A_{k-1} and B_{k-1}, one can show with a little bit of algebraic manipulation that the equation $f_c^k(z) = z$ for all z is equivalent to $-c_{k-2}(cz^2 + z - 1) = 0$ for all z, where $c_{k-2} = g_c^{k-2}(-1/c)$. This implies that $c_{k-2} = 0$. Thus, the set D_k is contained in the set of c values that make the equation $c_{k-2} = 0$ true. Since c_{k-2} is a rational function of c, there are only a finite number of solutions to this equation. So D_k is finite.

$$\begin{tikzcd} \mathbb{C}\cup\{\infty\} \arrow[r, dashed, "f_c"] \arrow[d, dashed, "T"'] & \mathbb{C}\cup\{\infty\} \arrow[d, dashed, "T"] \\ \mathbb{C}\cup\{\infty\} \arrow[r, dashed, "L_\alpha"] & \mathbb{C}\cup\{\infty\} \end{tikzcd}$$

FIGURE 1

We note that if c is a root of the equation, $c_{k-2}=0$, the sequences $\{f_c^{nk+j}(0)\}$, $j=0,1,\dots,k-1$ do not necessarily have k distinct limits.

III. Changing the setting The arguments used in Section II illustrate the power of complex analysis, but they are not very efficient. We will show how changing the setting of the problem results in a more concise proof of Theorem 2 and a more precise statement and proof of Theorem 3. In addition, we will be able to show that the union of the D_k's is dense on $(-\infty, -1/4)$. Our approach is motivated by a study of the work done by earlier mathematicians [**9**, p. 46–56] and more recently by [**3**, p. 1–48], [**5**, p. 1–60] and [**6**, p. 1–24, p. 57–74, p. 75–106].

For each $c=t^2+t$, $\mathrm{Re}(t)\geq -1/2$, $t\neq 0$, $t\neq -1/2$, $f_c(z)$ has exactly two fixed points, $1/(1+t)$ and $-(1/t)$. Let

$$T(z)=\frac{z+\dfrac{1}{t}}{z-\dfrac{1}{1+t}}. \tag{6}$$

Then,

$$T^{-1}(z)=\frac{\dfrac{1}{1+t}z+\dfrac{1}{t}}{z-1}. \tag{7}$$

Set $L_\alpha(z)=Tf_cT^{-1}(z)=\alpha z$, where $\alpha=(1+t)/-t$. FIGURE 1 displays the relationship among f_c, T, and L_α.

Note that $1/(1+t)$, $-(1/t)$, and $-(1/c)$ are mapped under $T(z)$ to the point at infinity, zero, and $-t/(1+t)$, respectively. Hence, instead of considering f_c, its iterates, and the parameter c-plane, we consider the linear map L_α, its iterates, and the parameter α-plane. We display the relationship between the two parameters in FIGURE 2. In the figure, $c=t^2+t$ is a one-to-one analytic map from $\{t\colon \mathrm{Re}(t)>-1/2\}$ onto $\mathbb{C}\setminus\{x\colon x\in\mathbb{R} \text{ and } x\leq -1/4\}$, and $\alpha=(1+t)/-t$ is a one-to-one analytic map of $\{t\colon \mathrm{Re}(t)>-1/2\}\setminus\{0\}$ onto the exterior of the unit disk. Thus, the relationship between α and c given by

$$c(\alpha)=\frac{1}{\alpha+1}\left(\frac{1}{\alpha+1}-1\right),\ |\alpha|>1,$$

is one-to-one and analytic. (Complex analysts will immediately observe that $c(\alpha)$ is the extremal function for the Bieberbach Conjecture [**8**, p. 189].) When $\mathrm{Re}(t)=-1/2$, $c=t^2+t$ maps $-1/2+iy$ and $-1/2-iy$ to $-1/4-y^2$, a point on the real line less than or equal to $-1/4$. Also, $\alpha=(1+t)/-t$ maps the line $\mathrm{Re}(t)=-1/2$ one-to-one and onto $\{|z|=1\}\setminus\{-1\}$. It follows that $c(\alpha)$ is a continuous, one-to-one map from the upper half (or lower half) of the unit circle onto the real numbers less than $-1/4$.

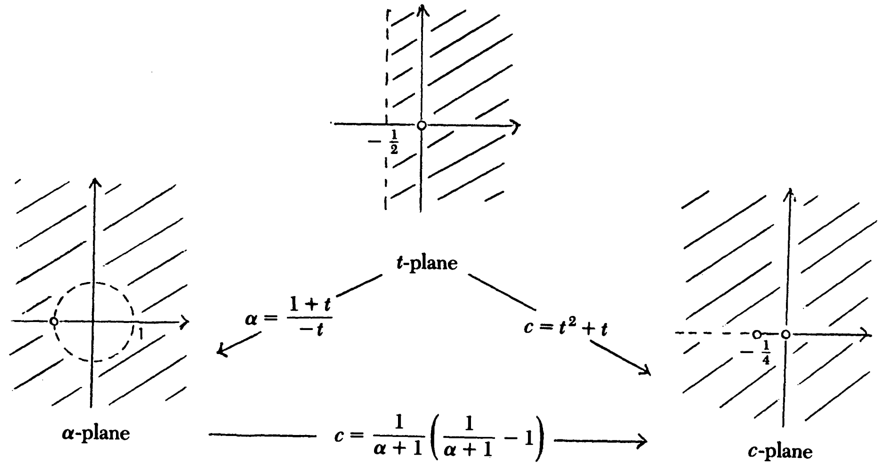

FIGURE 2

We now consider the iterates of $L_\alpha(z)$. If $c = t^2 + t$ where $\mathrm{Re}(t) > -1/2$ and $t \neq 0$, then $|\alpha| = |(1+t)/-t| > 1$. Hence, for all $z \in \mathbb{C}\setminus\{0\}$, $L_\alpha^n(z) = \alpha^n z$ converges to the point at infinity as $n \to \infty$. Since $f_c = T^{-1}L_\alpha T$, we have for all $z \in \mathbb{C}\setminus\{-(1/t)\}$, $f_c^n(z)$ converges to $1/(1+t)$ as $n \to \infty$. Since $0 \neq -(1/t)$, the sequence $\{f_c^n(0)\}$ converges. The cases where $t = -1/2$ and $t = 0$ are handled as before. *Thus, we have a concise proof of Theorem 2.*

Recall that the singularities of $f_c^n(z)$, $n = 1, 2, \ldots$, are the iterates of $-(1/c)$ under the inverse map $g_c(z)$. Under T, this corresponds to considering the iterates of $-t/(1+t)$ under the map $L_\alpha^{-1}(z) = (1/\alpha)z$. Clearly, these iterates converge to zero, and so the iterates of $g_c^n(-(1/c))$, $n = 1, 2, \ldots$, converge to $-(1/t)$.

In the case $c = t^2 + t$ where $\mathrm{Re}(t) = -1/2$ and $t \neq -1/2$, c is a point on the real line less than $-1/4$ and $|\alpha| = |(1+t)/-t| = 1$. We would like to determine those α on the unit disk such that $c(\alpha) \in D_k$. If $k \geq 3$, then we have $f_c^k(z) \equiv z$ as argued earlier and hence $L_\alpha^k(z) \equiv z$. Thus $\alpha^k = 1$ and so α is a kth root of unity. Under complex multiplication the kth roots of unity form a cyclic group with generator $\alpha_0 = e^{i(2\pi/k)}$, the principal kth root of unity. We denote this group by $G_k(\alpha_0)$. Using T and the definition of D_k, it follows that if $c(\alpha) \in D_k$, then the k subsequences $\{L_\alpha^{nk+j}((1+t)/-t)\}$ $j = 1, 2, \ldots, k$, each have distinct limits, which respectively are $L(\alpha) = \alpha^2$, $L^2(\alpha) = \alpha^3, \ldots, L^k(\alpha) = \alpha$. This implies that α must be a generator of $G_k(\alpha_0)$. We know [4, p. 71] that $\alpha = \alpha_0^s$ is a generator of $G_k(\alpha_0)$ if, and only if, $(s, k) = 1$. For a given k the number of such generators is $\varphi(k)$, where φ is the Euler totient function [4, p. 146]. Finally, we note that if α is a generator of $G_k(\alpha_0)$, then its inverse, which in this setting is the conjugate of α, is also a generator of $G_k(\alpha_0)$. Hence, there is an even number of generators of $G_k(\alpha_0)$, half of which are in the upper half-circle, the other half on the lower half-circle. We conclude that the generators of $G_k(\alpha_0)$ lying in the upper half-circle are mapped by $c(\alpha)$ one-to-one and onto D_k, and so the number of elements in D_k is given by $\varphi(k)/2$. We have a more precise proof of Theorem 3 that leads to the following result.

THEOREM 4. *The union of the D_k's is dense on $(-\infty, -1/4)$.*

Proof. Suppose $\delta > 0$ and let $\alpha = e^{i\theta}$ be any point in the upper half of the unit circle. Since the rationals are dense in $\mathbb{R}$, there exists a rational number m/k,

$(m, k) = 1$, such that $(\theta - \delta)/2\pi < m/k < (\theta + \delta)/2\pi$. This implies that $e^{i(2m\pi/k)}$ lies on the arc connecting $e^{i(\theta-\delta)}$ and $e^{i(\theta+\delta)}$ taken in the counterclockwise direction. Since $e^{i(2m\pi/k)}$ is a kth root of unity and a generator of $G_k(\alpha_0)$, we have that $\bigcup_{k=3}^{\infty} \{\alpha$ such that α is a generator of $G_k(\alpha_0)\}$ is dense on the unit circle.

Now let $\varepsilon > 0$ and c be such that the interval $(c - \varepsilon, c + \varepsilon)$ is contained in the interval $(-\infty, -1/4)$. Since $c(\alpha)$ is a one-to-one, continuous map of the upper half-circle onto the half-line, there exists an $\alpha = e^{i\theta}$ and $\delta > 0$ such that $c(\alpha) = c$, and the arc from $e^{i(\theta-\delta)}$ to $e^{i(\theta+\delta)}$ is mapped under $c(\alpha)$ into the interval $(c - \varepsilon, c + \varepsilon)$. Since we can find on the arc a kth root of unity that is a generator of the kth roots of unity, denoted α_k, its image, $c(\alpha_k)$, is a point of D_k in the interval. Hence, our theorem is established.

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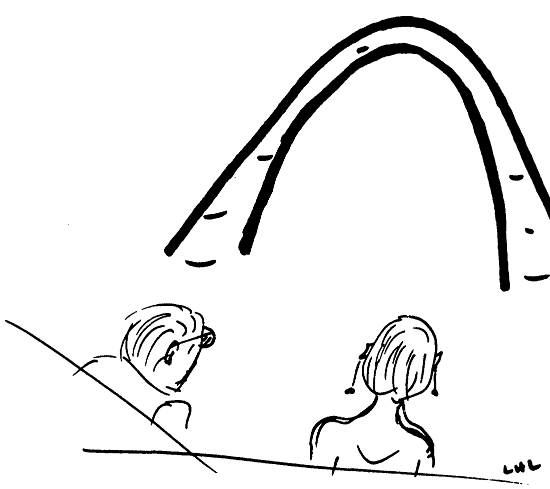
Two Mathematikers at the Gateway Arch in St. Louis

"*Cosh*, it's beautiful!"

"Yessiree. 1/2 $(e^{u} + e^{-u})$, for sure!"



—Lester H. Lange
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