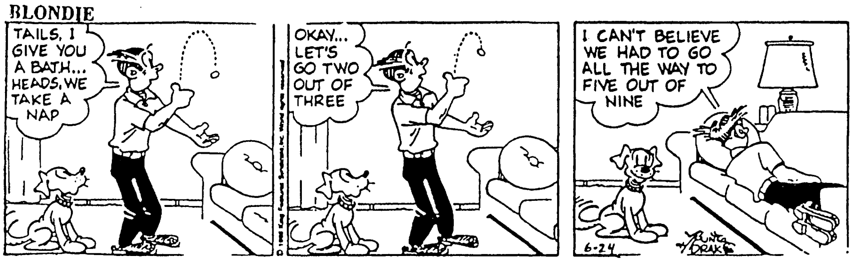
Math Bite: The Dagwood Random Nap



By tossing the coin only enough times for a majority of the outcomes to show heads, Dagwood has encountered a first-passage problem for random walk.

To formulate the random walk model corresponding to "steps" up and down with equal probabilities, assume $X_1, X_2, \ldots$ are obtained from independent and identically distributed Bernoulli trials, each with success probability $1/2$, so that $\text{Prob}(X_i = +1) = \text{Prob}(X_i = -1) = 1/2$. The corresponding (symmetric) simple random walk is then $S_n = X_1 + X_2 + \cdots + X_n$, $n = 1, 2, \ldots$.

Because S_n records the difference between the number of heads and the number of tails observed in the first n tosses, Dagwood's first passage problem then involves the random time $T_{\{1\}} = \min\{n \geq 1 | S_n = 1\}$. Observing that the event $\{S_n = 1\}$ can occur only for an odd number of tosses, we wish to calculate $\text{Prob}(T_{\{1\}} = 2n - 1)$ and $\text{Prob}(T_{\{1\}} > 2n - 1)$ for $n = 1, 2, \ldots$.

Explicit first-passage solutions are given for $n \geq 1$ by

$$\text{Prob}\left(T_{\{1\}} = 2n - 1\right) = \frac{1}{2n-1}\binom{2n-1}{n}\frac{1}{2^{2n-1}};$$

$$\text{Prob}\left(T_{\{1\}} > 2n - 1\right) = \binom{2n}{n}\frac{1}{2^{2n}}.$$

These formulas show that $\text{Prob}(T_{\{1\}} = 9) = 14 \times 1/2^9 \approx 0.027$ and $\text{Prob}(T_{\{1\}} > 9) = 252 \times 1/2^{10} \approx 0.246$. The (perhaps more relevant) second probability figure shows that nearly one in four Dagwood naps would require more than 9 coin tosses.

The first-passage probability solutions can be derived either from the analytic treatment in [**1**, p. 76–77] or the insightful geometric argument in [**2**]. Finally, it can be shown that a Dagwood random nap is indeed certain, but that the expected number of tosses is infinite [**1**, p. 272].

REFERENCES

1. W. Feller, *An Introduction to Probability Theory and Its Applications* (Vol. I, 3rd ed.), Wiley, New York, NY, 1968.
2. N. Hungerbühler, Proof without words: bijection between certain lattice paths, this MAGAZINE 71 (1998), 219.
3. Young and Drake, *Blondie*, King Features Syndicate, June 24, 1996.

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