False Conjecture Is True *ne*-Way

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Introduction In [**2**], Barry Kissane asks: How should a positive integer be partitioned into positive real summands whose product is as large as possible? For example, $13 = 5.5 + 7.5$, whose product is $5.5 \times 7.5 = 41.25$, but $13 = 5 + 4 + 4$, whose product $5 \times 4 \times 4 = 80$ is larger. If two summands differ, replacing them by their average will increase the product. Thus the product can be maximal only if the summands are equal. But how many summands should there be? Kissane conjectures: The maximum product occurs when all summands are equal and their common value is as close as possible to e. More precisely:

CONJECTURE. *For every positive integer k, the function $f(n) = \left(\frac{k}{n}\right)^n$, where n is a positive integer, is maximized when n minimizes $\left|\frac{k}{n} - e\right|$.*

While the conjecture holds for $k \le 52$, it fails for $k = 53$:

$$\left|e - \frac{53}{19}\right| \approx 0.071 > 0.068 \approx \left|e - \frac{53}{20}\right|,$$

but

$$\left(\frac{53}{19}\right)^{19} \approx 2.91691 \times 10^8 > 2.91687 \times 10^8 \approx \left(\frac{53}{20}\right)^{20}.$$

In [**3**], Eugene Krause notes the instructional value of such a "collector's item—a nontrivial proposition about positive integers that is true fifty-two times in a row before it fails." Indeed, the conjecture holds for *most* values of k. The only exceptions less than 10,000 are $k = 53$, 246, 439, and 632. I discovered that switching the n and the e inside the absolute value sign produces a (true) theorem:

THEOREM. *For every positive integer k, the function $f(n) = \left(\frac{k}{n}\right)^n$, where n is a positive integer, is maximized when n minimizes $\left|\frac{k}{e} - n\right|$.*

We will prove the theorem and then use it to study some counterexamples to the false conjecture. We will use the following facts about continued fractions (for proofs see, e.g., [**1**, [**4**], [**5**]).

Continued fractions Let $a_0, a_1, a_2, \ldots$ be positive integers. The continued fraction formed by the a_i, written in shorthand as $[a_0; a_1, a_2, a_3, \ldots, a_r]$ is the rational number

$$a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cdots + \frac{1}{a_r}}};$$

each numerator is 1. The a_i are called *partial quotients*. For example,

$$[1; 2, 3, 4] = 1 + \cfrac{1}{2 + \cfrac{1}{3 + \frac{1}{4}}} = \frac{43}{30}.$$

If $d \le r$, the rational number $[a_0; a_1, a_2, a_3, \dots, a_d]$ is called the d^{th} *convergent* to the continued fraction $[a_0; a_1, a_2, a_3, \dots, a_r]$. For example, $[1; \underbrace{1, 1, 1, 1, \dots, 1}_{r}]$ has convergents

$$[1] = 1,\ [1;1] = \frac{2}{1},\ [1;1,1] = \frac{3}{2},\ [1;1,1,1] = \frac{5}{3},\ [1;1,1,1,1] = \frac{8}{5}, \dots.$$

As one might guess from these results, the d^{th} convergent satisfies

$$\Big[1; \underbrace{1, 1, 1, 1, \dots, 1}_{d}\Big] = \frac{F_{d+2}}{F_{d+1}},$$

where F_d is the d^{th} Fibonacci number. An infinite continued fraction $[a_0; a_1, a_2, a_3, \dots]$ is defined as the limit of its convergents. For example,

$$[1; 1, 1, \dots] = \lim_{d \to \infty} \frac{F_{d+2}}{F_{d+1}} = \frac{1 + \sqrt{5}}{2}.$$

The continued fraction expansion of any positive number z can be obtained by setting $a_0 = \lfloor z \rfloor$ (where $\lfloor z \rfloor$ is the greatest integer not exceeding z) and iterating the function $f(t) = \frac{1}{t - \lfloor t \rfloor}$. If z is rational, the iteration eventually produces an integer, and the expansion is complete. We will use the following properties of continued fractions in what follows. We assume throughout that x is a given irrational number and $x = [a_0; a_1, a_2, a_3, \cdots]$ is its continued fraction.

Fact 1 The convergents $\frac{p_r}{q_r} = [a_0; a_1, a_2, a_3, \dots, a_r]$ to the continued fraction $[a_0; a_1, a_2, a_3, \dots]$ can be computed for $r \ge 0$ by the recursive formula

$$(p_{r+1}, q_{r+1}) = a_{r+1}(p_r, q_r) + (p_{r-1}, q_{r-1})$$

with initial conditions $(p_{-1}, q_{-1}) = (1, 0)$ and $(p_0, q_0) = (a_0, 1)$.

Fact 2 $e = [2; 1, 2, 1, 1, 4, 1, 1, 6, 1, 1, 8, 1, 1, 10, \dots]$. Using Fact 1 to compute the first few convergents to e (see table below) we see by induction that p_r is even and q_r is odd if and only if $r \equiv 0$ or $r \equiv 2 \pmod 6$.

i	0	1	2	3	4	5	6	7
$\frac{p_i}{q_i}$	$\frac{2}{1}$	$\frac{3}{1}$	$\frac{8}{3}$	$\frac{11}{4}$	$\frac{19}{7}$	$\frac{87}{32}$	$\frac{106}{39}$	$\frac{193}{71}$

Fact 3 Let a'_{r+1} denote $[a_{r+1}; a_{r+2}, a_{r+3}, \dots]$. Then

$$x - \frac{p_r}{q_r} = \frac{(-1)^r}{q_r^2} \frac{1}{a'_{r+1} + \frac{q_{r-1}}{q_r}}.$$

It follows that the convergents p_n/q_n lie alternately above and below x.

Fact 4 If $\left| x - \frac{p}{q} \right| < \frac{1}{2q^2}$, for relatively prime integers p and q, then p/q is a convergent to the continued fraction of x.

We now use these facts to study rational approximations of e.

LEMMA 1. *Let p_r/q_r be the rth convergent to the continued fraction for e. If $r \equiv 0$ or $r \equiv 2 \pmod 6$, then*

$$q_r |q_r e - p_r| \ge \frac{6}{13}.$$

Proof. By Fact 3, it suffices to show $a'_{r+1} + \frac{q_{r-1}}{q_r} < \frac{13}{6}$. If $r = 6j$, then

$$a'_{r+1} = [1; 4j+2, 1, 1, \dots] < 1 + \frac{1}{4j+2} \le \frac{7}{6},$$

so

$$a'_{r+1} + \frac{q_{r-1}}{q_r} < \frac{7}{6} + 1 = \frac{13}{6}.$$

If $r = 6j + 2$, then $a'_{r+1} = [1; 1, 4j+4, 1, \dots] < 2$. By Facts 1 and 2,

$$\frac{q_{r-1}}{q_r} = \frac{q_{r-1}}{(4j+2)q_{r-1} + q_{r-2}} < \frac{1}{4j+2} \le \frac{1}{6},$$

so

$$a'_{r+1} + \frac{q_{r-1}}{q_r} < \frac{13}{6}. \qquad \blacksquare$$

LEMMA 2. *For all* $t > 0$, $\dfrac{(t+1)^{t+1}}{t^t} \le e\left(t + \dfrac{1}{2}\right)$.

Proof. If $h(x)$ is a concave function, integrable on $[a, b]$, then Jensen's inequality for integrals gives $h\left(\dfrac{a+b}{2}\right) \ge \dfrac{1}{b-a}\displaystyle\int_a^b h(x)\,dx$. Take $h(x) = \ln x$ and $[a, b] = [t, t+1]$ to get

$$\ln\left(t + \frac{1}{2}\right) \ge (t+1)\ln(t+1) - t\ln t - 1,$$

which is equivalent to the lemma. $\blacksquare$

Proof of the Theorem Recall that $f(n) = \left(\frac{k}{n}\right)^n$. By examining the first derivative of $\ln f(n)$, we see that the maximal product occurs either at $n = \left\lfloor \frac{k}{e} \right\rfloor$ or at $\left\lfloor \frac{k}{e} \right\rfloor + 1$. Let $t = \left\lfloor \frac{k}{e} \right\rfloor$. Direct calculation verifies the theorem for $k \le 71$, so we may assume that $k > 71$, or, equivalently, that $t \ge 27$. If $\frac{k}{e} > t + \frac{1}{2}$ then $\left|\frac{k}{e} - n\right|$ is minimized when $n = t + 1$. By Lemma 2,

$$k > e\left(t + \tfrac{1}{2}\right) > \frac{(t+1)^{t+1}}{t^t},$$

which implies that $f(t+1) > f(t)$, as desired.

Now suppose that $\frac{k}{e} < t + \frac{1}{2}$. Let $s = t + \frac{1}{2} - \frac{k}{e} > 0$. By Fact 2 and Lemma 3,

$$\frac{6}{13} \le q_n|q_n e - p_n| = (2t+1)|(2t+1)e - 2k| = 2e(2t+1)s.$$

Since $t \ge 27$, we have $s \ge \frac{3}{13e(2t+1)} > \frac{1}{24t}$, and so $k < \left(t + \frac{1}{2} - \frac{1}{24t}\right)e$. Dividing by $t + 1$,

$$\frac{k}{t+1} < e\left(\frac{t + \frac{1}{2} - \frac{1}{24t}}{t+1}\right) = e\left(1 - \frac{12t+1}{24t(t+1)}\right).$$

Taking logs and writing $\ln(1+z)$ as an alternating series gives

$$\ln\frac{k}{t+1} < 1 + \ln\left(1 - \frac{12t+1}{24t(t+1)}\right) = 1 - \sum_{k=1}^{\infty} \frac{1}{k}\left(\frac{12t+1}{24t(t+1)}\right)^k.$$

Therefore,

$$t\ln\left(1+\frac{1}{t}\right)-\ln\frac{k}{t+1} > t\sum_{k=1}^{4}\frac{(-1)^{k-1}}{kt^k}-1+\sum_{k=1}^{3}\frac{1}{k}\left(\frac{12t+1}{24t(t+1)}\right)^k$$
$$=\frac{864t^3-7308t^2-17208t-10367}{41472t^3(t+1)^3}.$$

The numerator has only one real root, at $t \approx 10.47$, and is therefore positive since $t \geq 27$. Thus $f(t) > f(t+1)$. ■

Counterexamples to the conjecture A computer search found the first twelve counterexamples to the conjecture: 53, 246, 439, 632, 12973, 62144, 111315, 160486, 209657, 258828, 7332553, 205052656.

In fact, there are infinitely many counterexamples. Suppose k is a counterexample, and let n be the integer that minimizes $|\frac{k}{n}-e|$. Then e must lie between $\frac{k}{n}$ and $\frac{k}{n+1}$ or between $\frac{k}{n-1}$ and $\frac{k}{n}$. If $\frac{k}{n+1} < e < \frac{k}{n}$, then $\frac{2e}{k} > \frac{2n+1}{n(n+1)}$. Since n does not minimize $|\frac{k}{e}-n|$, $\frac{k}{e}$ is closer to $n+1$ than to n, whence $2\frac{k}{e} > 2n+1$. Multiplying these last two inequalities yields $4 > \frac{(2n+1)^2}{n^2+n}$, a contradiction. Hence $\frac{k}{n} < e < \frac{k}{n-1}$. In this case, $\frac{2e}{k} > \frac{2n-1}{n(n-1)}$ and $\frac{2k}{e} < 2n-1$, so k is a counterexample if and only if

$$0 < e-\frac{2k}{2n-1} < \frac{e}{(2n-1)^2}. \tag{1}$$

By Fact 3, the right-hand inequality in (1) is satisfied by each convergent p_r/q_r to e. By Fact 2, there are integers k and n with $\frac{p_r}{q_r}=\frac{2k}{2n-1}$ if $r \equiv 0$ or $r \equiv 2 \pmod 6$. By Fact 3, the successive convergents must be alternately above and below e. Since $p_0/q_0 = 2/1 < e$, we have $p_r/q_r < e$ for all even r. Hence (1) holds for $2k = p_r$ and $2n-1 = q_r$, where $r \equiv 0$ or $r \equiv 2 \pmod 6$. This method produces infinitely many counterexamples of the form $k = p_{6j}/2$ or $k = p_{6j+2}/2$, for $j = 1, 2, 3, \ldots$. These account for six of the first twelve counterexamples, mentioned earlier:

j	1	2	3
$p_{6j}/2$	53	12973	7332553
$p_{6j+2}/2$	632	258828	205052656

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