Shortest Shoelaces

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Introduction Halton [1] first studied the question of finding the shortest possible lacing of a shoe. Misiurewicz [2] generalized Halton's original result to handle irregularly placed eyelets. We generalize Halton's result in a different direction by considering lacings that do not necessarily alternate in a regular way.

Halton proved that the American style lacing is the shortest among all possible alternating lacings. However, some common lacings are not covered by Halton's definition. Neither the *ice skater's lacing* (FIGURE 1A) nor the *playground lacing* (FIGURE 1B) is alternating.

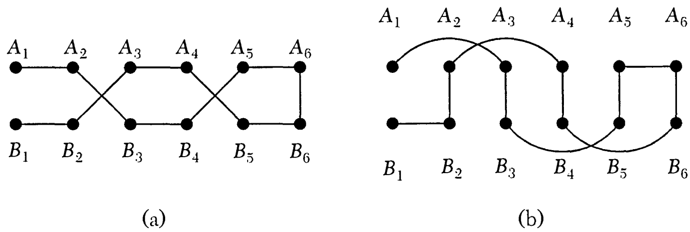

FIGURE 1
(a) ice skater's lacing, (b) playground lacing

Something to avoid in sensible lacings is the occurrence of three consecutive eyelets on the same side. In this case, why bother to use the middle one? This condition is equivalent to requiring that every eyelet (except possibly the first and last) be the endpoint of at least one crossing.

A *lacing* of degree n is an ordering of the set

$$\{A_1, A_2, \dots, A_n, B_1, B_2, \dots, B_n\}$$

starting with A_1 and ending with B_1, such that no more than two A's (respectively B's) occur in consecutive places. A *bipartite lacing* of degree n is a lacing of degree n in which the A's and B's alternate. Halton's definition is equivalent to our definition of bipartite lacings.

The main result Following Halton, we shall assume that the two rows of eyelets are parallel and that the eyelets in each row are evenly spaced. A simple calculation shows that the ice skater's lacing is significantly shorter than the American style lacing. In fact, the ice skater's lacing is minimal. A modification of Halton's original proof for bipartite lacings [1] demonstrates this fact.

THEOREM. *For any n, L_{IS} is a shortest lacing of degree n.*

The idea of the proof is as follows. Start with a lacing L. Following Halton, create a path P in a rectangular grid so that the path starts in the upper left corner and so that all the segments of P are horizontal, vertical, or diagonal downward and to the right. FIGURE 2 illustrates this transformation in two cases. The dashed line represents L_{IS}, while the solid line represents another particular lacing.

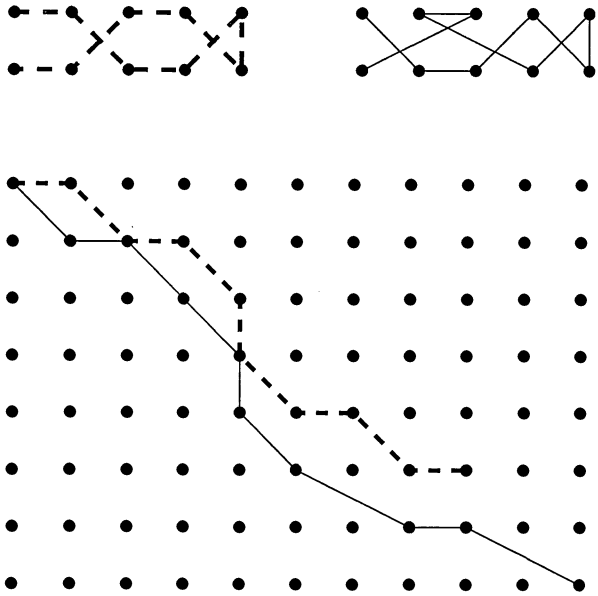

FIGURE 2
At top, the iceskater's lacing and another lacing. At bottom, these lacings are unwound.

We do not know exactly where P ends. The original ice skater's lacing has as many non-crossing steps as possible. Hence P_{IS} ends somewhere above (or possibly in the same row as) the endpoint of P.

Moreover, the horizontal length of P must be at least $2n-2$ units since L connects A_1 to A_n to B_1. The horizontal length of P_{IS} is exactly $2n-2$ units, so we know that P must end to the right of (or possibly in the same column as) the endpoint of P_{IS}.

From this point, slightly technical but straightforward arguments, which we leave as an exercise for the reader, show that P is no shorter than P_{IS}. One way to show this is to lengthen P_{IS} by replacing horizontal segments with diagonal segments until P and P_{IS} end in the same row. Then cancel horizontal and vertical segments of equal length from P_{IS} and P until P_{IS} becomes a straight line. This completes the proof.

Uniqueness For n even, L_{IS} is the *unique* shortest lacing. When n is odd, there are exactly $(n+1)/2$ shortest lacings. They differ only by reordering the horizontal and crossing segments.

Now you know the best way of temporarily relacing your shoe the next time your shoelace breaks!

REFERENCES

1. John H. Halton, The shoelace problem, *Math. Intelligencer* 17 (no. 4) (1995), 36–41.
2. Michal Misiurewicz, Lacing irregular shoes, *Math. Intelligencer* 18 (no. 4) (1996), 32–34.