Partitions into Consecutive Parts

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It is known, though perhaps not as well as it should be, that the number of partitions of \( n \) into (one or more) consecutive parts is equal to the number of odd divisors of \( n \). (This is the special case \( k = 1 \) of a theorem of J. J. Sylvester [1, §46], to the effect that the number of partitions of \( n \) into distinct parts with \( k \) sequences of consecutive parts is equal to the number of partitions of \( n \) into odd parts (repetitions allowed) precisely \( k \) of which are distinct.)

For instance,

\[
15 = 7 + 8 = 4 + 5 + 6 = 1 + 2 + 3 + 4 + 5,
\]

so 15 has four partitions into consecutive parts, and 15 has four odd divisors, 1, 3, 5, and 15.

We shall prove the following result.

**Theorem.** The number of partitions of \( n \) into an odd number of consecutive parts is equal to the number of odd divisors of \( n \) less than \( \sqrt{2n} \), while the number of partitions into an even number of consecutive parts is equal to the number of odd divisors greater than \( \sqrt{2n} \).

**Proof.** Suppose \( n \) is the sum of an odd number of consecutive parts. Then the middle part is an integer and is the average of the parts. Suppose the middle part is \( a \), and the number of parts is \( 2k + 1 \). The partition of \( n \) is

\[
n = (a - k) + \cdots + a + \cdots + (a + k)
\]

and \( n = (2k + 1)a \). So \( d = 2k + 1 \) is an odd divisor of \( n \) and its codivisor is \( d' = a \). Note that \( a - k \geq 1 \), that is, \( 2a - (2k + 1) > 0, d < 2d', d < 2n/d, \) and \( d^2 < 2n \). Conversely, suppose \( d \) is an odd divisor of \( n \) with \( d^2 < 2n \), and codivisor \( d' \). Then \( d < 2d' \), and if we write \( 2k + 1 = d, a = d' \) then

\[
n = (a - k) + \cdots + a + \cdots + (a + k)
\]

is a partition of \( n \) into \( 2k + 1 \) consecutive parts.

Next, suppose \( n \) is the sum of an even number, \( 2k \), of consecutive parts. Then the average part is \( a + 1/2 \) for some integer \( a \), the partition of \( n \) is

\[
n = (a + 1 - k) + \cdots + a + (a + 1) + \cdots + (a + k),
\]

and \( n = 2k(a + 1/2) = k(2a + 1) \). Then \( d = 2a + 1 \) is an odd divisor of \( n \) and its codivisor is \( d' = k \). Note that \( a - k \geq 0 \), \( (2a + 1) - 2k > 0, d > 2d', d > 2n/d, \) and \( d^2 > 2n \).
Conversely, suppose \( d \) is an odd divisor of \( n \) with \( d^2 > 2n \), with codivisor \( d' \). Then \( d > 2d' \), and if we write \( 2a + 1 = d, k = d' \), then

\[
n = (a + 1 - k) + \cdots + a + (a + 1) + \cdots + (a + k)
\]

is a partition of \( n \) into an even number of consecutive parts. Q.E.D.

REFERENCE

Means Generated by an Integral

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For a pair of distinct positive numbers, \( a \) and \( b \), a number of different expressions are known as means:

1. the arithmetic mean: \( A(a, b) = (a + b)/2 \)
2. the geometric mean: \( G(a, b) = \sqrt{ab} \)
3. the harmonic mean: \( H(a, b) = 2ab/(a + b) \)
4. the logarithmic mean: \( L(a, b) = (b – a)/(\ln b - \ln a) \)
5. the Heronian mean: \( N(a, b) = (a + \sqrt{ab} + b)/3 \)
6. the centroidal mean: \( T(a, b) = 2(a^2 + ab + b^2)/3(a + b) \)

Recently, Professor Howard Eves [1] showed how many of these means occur in geometrical figures. The integral in our title is

\[
f(t) = \frac{\int_a^b x^{t+1} \, dx}{\int_a^b x^t \, dx}, \tag{1}
\]

which encompasses all these means: particular values of \( t \) in (1) give each of the means on our list. Indeed, it is easy to verify that

\[
f(-3) = H(a, b), \quad f\left(-\frac{3}{2}\right) = G(a, b), \quad f(-1) = L(a, b),
\]
\[
f\left(-\frac{1}{2}\right) = N(a, b), \quad f(0) = A(a, b), \quad f(1) = T(a, b).
\]

Moreover, upon showing that \( f(t) \) is strictly increasing, we can conclude that

\[
H(a, b) \leq G(a, b) \leq L(a, b) \leq N(a, b) \leq A(a, b) \leq T(a, b), \tag{2}
\]

with equality if and only if \( a = b \).