Partitions into Consecutive Parts

M. D. HIRSCHHORN
University of New South Wales
Sydney 2052, Australia
m.hirschhorn@unsw.edu.au

P. M. HIRSCHHORN
5/312 Finchley Road, Hampstead
London NW3 7AG, U.K.
phirschhorn.mba2005@london.edu

It is known, though perhaps not as well as it should be, that the number of partitions of n into (one or more) consecutive parts is equal to the number of odd divisors of n. (This is the special case $k = 1$ of a theorem of J. J. Sylvester [**1**, §46], to the effect that the number of partitions of n into distinct parts with k sequences of consecutive parts is equal to the number of partitions of n into odd parts (repetitions allowed) precisely k of which are distinct.)

For instance,

$$15 = 7 + 8 = 4 + 5 + 6 = 1 + 2 + 3 + 4 + 5,$$

so 15 has four partitions into consecutive parts, and 15 has four odd divisors, 1, 3, 5, and 15.

We shall prove the following result.

THEOREM. *The number of partitions of n into an odd number of consecutive parts is equal to the number of odd divisors of n less than $\sqrt{2n}$, while the number of partitions into an even number of consecutive parts is equal to the number of odd divisors greater than $\sqrt{2n}$.*

Proof. Suppose n is the sum of an odd number of consecutive parts. Then the middle part is an integer and is the average of the parts. Suppose the middle part is a, and the number of parts is $2k + 1$. The partition of n is

$$n = (a - k) + \cdots + a + \cdots + (a + k)$$

and $n = (2k + 1)a$. So $d = 2k + 1$ is an odd divisor of n and its codivisor is $d' = a$. Note that $a - k \geq 1$, that is, $2a - (2k + 1) > 0$, $d < 2d'$, $d < 2n/d$, and $d^2 < 2n$. Conversely, suppose d is an odd divisor of n with $d^2 < 2n$, and codivisor d'. Then $d < 2d'$, and if we write $2k + 1 = d$, $a = d'$ then

$$n = (a - k) + \cdots + a + \cdots + (a + k)$$

is a partition of n into $2k + 1$ consecutive parts.

Next, suppose n is the sum of an even number, $2k$, of consecutive parts. Then the average part is $a + 1/2$ for some integer a, the partition of n is

$$n = (a + 1 - k) + \cdots + a + (a + 1) + \cdots + (a + k),$$

and $n = 2k(a + 1/2) = k(2a + 1)$. Then $d = 2a + 1$ is an odd divisor of n and its codivisor is $d' = k$. Note that $a - k \geq 0$, $(2a + 1) - 2k > 0$, $d > 2d'$, $d > 2n/d$, and $d^2 > 2n$.

Conversely, suppose d is an odd divisor of n with $d^2 > 2n$, with codivisor d'. Then $d > 2d'$, and if we write $2a + 1 = d$, $k = d'$, then

$$n = (a + 1 - k) + \cdots + a + (a + 1) + \cdots + (a + k)$$

is a partition of n into an even number of consecutive parts. ■

REFERENCE

1. J. J. Sylvester, A constructive theory of partitions, arranged in three acts, an interact and an exodion, *Amer. J. Math.* **5** (1882), 251–330.

Means Generated by an Integral

HONGWEI CHEN
Department of Mathematics
Christopher Newport University
Newport News, VA 23606
hchen@cnu.edu

For a pair of distinct positive numbers, a and b, a number of different expressions are known as *means*:

1. the arithmetic mean: $A(a, b) = (a + b)/2$
2. the geometric mean: $G(a, b) = \sqrt{ab}$
3. the harmonic mean: $H(a, b) = 2ab/(a + b)$
4. the logarithmic mean: $L(a, b) = (b - a)/(\ln b - \ln a)$
5. the Heronian mean: $N(a, b) = (a + \sqrt{ab} + b)/3$
6. the centroidal mean: $T(a, b) = 2(a^2 + ab + b^2)/3(a + b)$

Recently, Professor Howard Eves [1] showed how many of these means occur in geometrical figures. The integral in our title is

$$f(t) = \frac{\int_a^b x^{t+1}\,dx}{\int_a^b x^t\,dx}, \tag{1}$$

which encompasses all these means: particular values of t in (1) give each of the means on our list. Indeed, it is easy to verify that

$$f(-3) = H(a, b), \qquad f\left(-\frac{3}{2}\right) = G(a, b), \qquad f(-1) = L(a, b),$$
$$f\left(-\frac{1}{2}\right) = N(a, b), \qquad f(0) = A(a, b), \qquad f(1) = T(a, b).$$

Moreover, upon showing that $f(t)$ is strictly increasing, we can conclude that

$$H(a, b) \le G(a, b) \le L(a, b) \le N(a, b) \le A(a, b) \le T(a, b), \tag{2}$$

with equality if and only if $a = b$.