Running with Rover

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It all started one day when I went running on a trail with my faithful dog Rover. Now Rover does not actually rove. In fact, Rover is so well trained that he always runs exactly one yard to my right. As long as I change direction smoothly, he will adjust his speed and his path perfectly so as to remain in this position. Of course, I must choose my path so that he is not required to run through trees or other obstacles.

On this particular day our trail was flat but curvy. We looped around several times, as shown in FIGURE 1. As nearly as I can tell from a map, the farthest we got from our starting and finishing point S was about a mile as the crow flies (not as the Crofoot runs).

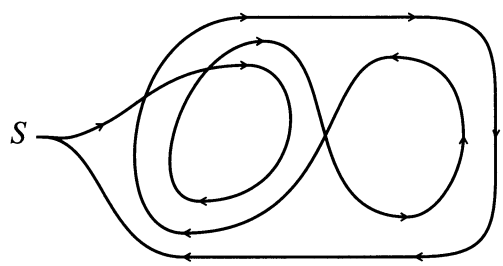

Figure 1 The trail

Prostrate on the couch at the end of our run, I could not help but notice that Rover still had lots of energy. This upset my athletic self-image to the extent that I almost resolved to cut back on ice cream and begin a rigorous program of monthly exercise. Fortunately, I was saved from this fanaticism by the reassuring realization that I had run farther than Rover, having been on the outside on most of our turns.

Now I would like to know this: How much farther did I run than Rover?

Surprisingly, the exact answer to this question can be determined from FIGURE 1 without any additional information. In the course of arriving at that answer, we will develop some ideas that are central to differential geometry and topology. (Expert readers who already know the answer may be interested to know that this elementary treatment does not require using arc length as a new parameter. Switching between two independent variables, time and arc length, is often a stumbling block for students.)

Since FIGURE 1 is quite complicated, let's consider a simpler situation first. Suppose that I run with Rover on a track instead of a trail. The track has the usual shape, with semicircular ends connected by straight sides. As we round the ends, the outside runner follows a circle of radius $R + 1$ while the inside runner follows a circle of radius R. During one complete lap, the difference in total distance is $2\pi(R + 1) - 2\pi R = 2\pi$ yards. Note that this difference does not involve R. So the length of the track does not matter.

Now we start to wonder (you and I, that is—not Rover!): since the length of the track does not matter, perhaps the shape of the track does not matter. The difference in distance will be greatest along stretches where the track turns most quickly, but it seems plausible that the difference in total distance depends only on the total amount of turning that occurs along the way, taking account of the direction of the turns. We will show that this is indeed the case.

The central idea is that smooth curves can be approximated by arcs of circles. What this means physically is that when I am running on a curved path, both my direction and the sharpness of my turn at any moment are the same as if I were running on some circular path of an appropriate radius. This approximating circle at a point on my path is called the *osculating circle* at the point, and its radius is the *radius of curvature* of my path at the point. As I run with Rover along a curved trail, the difference in our speeds at any time is related to the different radii of our osculating circles. In order to maintain his position alongside of me, Rover must adjust his speed and direction to ensure that at each instant his osculating circle has a radius one yard larger or smaller than mine, depending on the direction of our turn.

Curves and angle functions The position of an object moving in a plane (a running person or dog, for example) can be described relative to some fixed reference point O by a vector-valued function $\mathbf{r}(t) = \langle x(t), y(t) \rangle$, where t is time. This function is a *parametrization* for an *oriented curve* Γ. FIGURE 2 shows such a curve, parametrized by $\mathbf{r}(t)$ as t varies within an interval $[a, b]$. The arrows on the curve indicate the direction along the curve corresponding to increasing t.

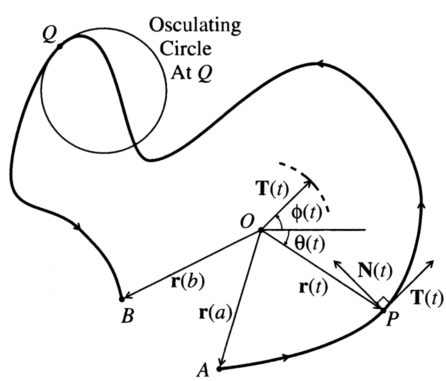

Figure 2 Analysis of a Curve

If $\mathbf{r}(t)$ is differentiable, the derivative $\mathbf{r}'(t) = \langle x'(t), y'(t) \rangle$ represents the *velocity* of an object whose position is given by $\mathbf{r}(t)$. For any t such that $\mathbf{r}'(t) \neq \mathbf{0}$, the vector $\mathbf{r}'(t)$ is tangent to Γ at the point $(x(t), y(t))$ and points in the direction of Γ. The magnitude of this vector is the speed, which can be integrated to calculate the distance travelled along Γ.

The vector $\langle -y'(t), x'(t) \rangle$, obtained by rotating the velocity vector counter-clockwise 90 degrees, is normal to Γ at the same point. Dividing by the magnitudes of these vectors, we obtain a *unit tangent vector* $\mathbf{T}(t)$ and a *unit normal vector* $\mathbf{N}(t)$:

$$\mathbf{T}(t) = \frac{\mathbf{r}'(t)}{|\,\mathbf{r}'(t)\,|} = \frac{\langle x'(t), y'(t) \rangle}{|\,\mathbf{r}'(t)\,|}, \tag{1}$$

$$\mathbf{N}(t) = \frac{\langle -y'(t), x'(t) \rangle}{|\,\mathbf{r}'(t)\,|}. \tag{2}$$

These vectors are shown at the point P in FIGURE 2.

Let Γ be a curve parametrized by a continuous function $\mathbf{r}(t)$. Assume that the origin does not lie on Γ, so that $\mathbf{r}(t)$ is never the zero vector. For any particular value of t, the point $(x(t), y(t))$ on Γ can be represented using polar coordinates $(r(t), \theta(t))$, where

$$r(t) = \sqrt{[x(t)]^2 + [y(t)]^2} = |\,\mathbf{r}(t)\,|, \tag{3}$$

$$\cos\theta(t) = \frac{x(t)}{r(t)}, \qquad \sin\theta(t) = \frac{y(t)}{r(t)}. \tag{4}$$

The angle $\theta(t)$ is not uniquely defined by these equations. If $\theta_0(t)$ is one solution of the equations, then the general solution is $\theta(t) = \theta_0(t) + 2\pi n(t)$, where $n(t)$ is any integer-valued function of t. However, once we have selected an angle from among the various possible angles at any particular point on the curve, we can change this angle continuously as we move along the curve, never jumping by a multiple of 2π. The resulting *continuous angle function* $\theta(t)$ is not uniquely determined by the parametrization $\mathbf{r}(t)$, but any two continuous angle functions differ by a constant, which is an integer multiple of 2π. This informal discussion of continuous angle functions is made rigorous elsewhere [**1**; **3**, pp. 17–19, 35–37]. From now on, whenever we talk about an angle function for a curve, we will always mean a continuous angle function—a continuous function $\theta(t)$ such that equations (4) are satisfied for all t.

As t varies from $t = t_1$ to $t = t_2$, the change $\theta(t_2) - \theta(t_1)$ does not depend on the particular angle function $\theta(t)$, since any two angle functions differ by a constant. This change can be interpreted as the total angle through which the position vector $\mathbf{r}(t)$ turns as t varies from t_1 to t_2. In particular, $\theta(b) - \theta(a)$ represents the total angle through which $\mathbf{r}(t)$ turns along the entire curve Γ. By the total angle we mean the total *signed* angle, which increases as $\mathbf{r}(t)$ turns counterclockwise and decreases as $\mathbf{r}(t)$ turns clockwise, possibly taking negative values.

If $\mathbf{r}(a) = \mathbf{r}(b)$, then Γ is a *closed* curve. In this case the change $\theta(b) - \theta(a)$ must be an integer multiple of 2π because the angles $\theta(a)$ and $\theta(b)$ refer to the same point. The integer $[\theta(b) - \theta(a)]/(2\pi)$ represents the number of times that Γ winds around the origin (with counterclockwise counting as positive and clockwise counting as negative). This integer is called the *index*, or *winding number*, of Γ with respect to the origin.

If $\mathbf{r}(t)$ is differentiable, then any angle function $\theta(t)$ has a derivative, which can be calculated from equations (3) and (4):

$$\theta'(t) = \frac{x(t)y'(t) - x'(t)y(t)}{|\,\mathbf{r}(t)\,|^2} = \frac{x(t)y'(t) - x'(t)y(t)}{[x(t)]^2 + [y(t)]^2}. \tag{5}$$

The change in angle along the curve can be computed by integration:

$$\theta(b) - \theta(a) = \int_a^b \theta'(t)\,dt.$$

Having talked about angle functions, we should give some indication of how we intend to apply them. An angle function for a curve describes the turning of the position vector $\mathbf{r}(t)$. For the curve in FIGURE 1, what concerns us is not the turning of $\mathbf{r}(t)$ but the turning of a *tangent vector*. Therefore, in order to apply the idea of an angle function, we will need to consider a curve constructed from the given curve by using the tangent vector $\mathbf{T}(t)$ of the given curve as the parametrization for the new curve. This idea is developed in the next section.

Turning rate along a curve The unit tangent vector $\mathbf{T}(t)$ is defined at all points on Γ where the velocity $\mathbf{r}'(t)$ is not zero. We now demand that $\mathbf{T}(t)$ be a continuous function defined on the entire parameter interval $[a, b]$. This will happen when the following two conditions are satisfied:

1. The function $\mathbf{r}(t)$ is continuously differentiable on $[a, b]$ (using one-sided derivatives at the endpoints a and b),
2. At any point where $\mathbf{r}'(t) = 0$, there is no change in direction of the curve, so that the function $\mathbf{T}(t)$ can be continuously extended to include this point in its domain.

Then $\mathbf{T}(t)$ may be regarded as a parametrization for a curve Γ' called the *tangent indicatrix*. The points of Γ' all lie on the unit circle. The vector $\mathbf{T}(t)$ may trace out parts of this circle repeatedly as t varies, in which case these repetitions are considered to be distinct parts of the parametrized curve Γ'.

Let $\phi(t)$ denote a continuous angle function for Γ'. We want to apply equation (5) to obtain a formula for $\phi'(t)$. The position vector for Γ' is $\mathbf{T}(t) = c(t)\mathbf{r}'(t)$, where $c(t) = 1/|\mathbf{r}'(t)|$. Therefore, in equation (5) we will replace $x(t)$ by $c(t)x'(t)$ and $y(t)$ by $c(t)y'(t)$). Assuming that $x''(t)$ and $y''(t)$ both exist for all t, we calculate

$$\phi'(t) = \frac{x'(t)y''(t) - x''(t)y'(t)}{|\mathbf{r}'(t)|^2}. \tag{6}$$

Actually this formula can be derived assuming only that $c(t)$ is *any* positive, differentiable function, not necessarily $1/|\mathbf{r}'(t)|$. Intuitively this is because the amount of turning of a tangent vector does not depend on the length of the tangent vector. For purposes of defining the angle function $\phi(t)$, we could just as well use a curve parametrized simply by $\mathbf{r}'(t)$ instead of $\mathbf{T}(t)$.

A straightforward calculation, starting from equations (1) and (2) and using (6), yields the following formulas for the derivatives of the unit tangent vector and the unit normal vector:

$$\mathbf{T}'(t) = \phi'(t)\mathbf{N}(t), \qquad \mathbf{N}'(t) = -\phi'(t)\mathbf{T}(t). \tag{7}$$

Taking magnitudes here, and recalling that $\mathbf{T}(t)$ and $\mathbf{N}(t)$ are unit vectors, we see that $|\mathbf{T}'(t)| = |\mathbf{N}'(t)| = |\phi'(t)|$. Thus the magnitude of the rate of change of the angle $\phi(t)$ is the same as the magnitude of the rate of change of the unit tangent vector and the unit normal vector. All three of these rates are equivalent measures of the rate of turning along Γ.

The rate of turning is related to the radius of curvature. Most calculus books derive the following expression for the radius of curvature at any point $(x(t), y(t))$ on Γ:

$$\frac{([x'(t)]^2 + [y'(t)]^2)^{3/2}}{|x'(t)y''(t) - x''(t)y'(t)|}.$$

Using equation (6), we can express this as $|\mathbf{r}'(t)|/|\phi'(t)|$. The denominator here may be zero for some particular value of t, in which case the radius of curvature at the corresponding point on the curve Γ is considered to be either undefined or infinite. This happens at all points if the curve is a straight line. It also happens at points where the curve changes turning direction, and, for an instant, is not curving at all.

If we omit the absolute value in the denominator in the above expressions for the radius of curvature, we obtain a *signed* radius of curvature, which is positive when the curve is turning counterclockwise and negative when it is turning clockwise. Letting $R(t)$ denote the signed radius of curvature at the point with position vector $\mathbf{r}(t)$, we have

$$R(t) = \frac{|\mathbf{r}'(t)|}{\phi'(t)}. \tag{8}$$

Back to Rover Now consider Rover and me as we run along the trail shown in FIGURE 1. Choose a convenient coordinate system, and let $\mathbf{r}_1(t)$ be a parametrization for my path. Define unit tangent and unit normal vectors as before, $\mathbf{T}(t)$ and $\mathbf{N}(t)$.

Assuming that Rover runs exactly α yards to my right, where α is a constant, Rover's path is parametrized by

$$\mathbf{r}_2(t) = \mathbf{r}_1(t) - \alpha\mathbf{N}(t).$$

Taking the derivative and using equations (1) and (7), we get

$$\begin{aligned}\mathbf{r}_2'(t) &= \mathbf{r}_1'(t) - \alpha\mathbf{N}'(t) = \mathbf{r}_1'(t) + \alpha\phi'(t)\mathbf{T}(t)\\ &= [\,|\,\mathbf{r}_1'(t)\,| + \alpha\phi'(t)]\mathbf{T}(t).\end{aligned}$$

Then, since $\mathbf{T}(t)$ is a unit vector,

$$|\,\mathbf{r}_2'(t)\,| = \big|\,|\,\mathbf{r}_1'(t)| + \alpha\phi'(t)\,\big| = |\,\mathbf{r}_1'(t)\,| + \alpha\phi'(t),$$

provided the quantity $|\,\mathbf{r}_1'(t)\,| + \alpha\phi'(t)$ is never negative. This is a reasonable assumption, as we will see shortly. First we will complete our calculation. Using this equation and the standard formula for arc length, we calculate the difference in arc length along the two paths:

$$\begin{aligned}L_2 - L_1 &= \int_a^b |\,\mathbf{r}_2'(t)\,|\,dt - \int_a^b |\,\mathbf{r}_1'(t)\,|\,dt\\ &= \alpha\int_a^b \phi'(t)\,dt = \alpha[\phi(b) - \phi(a)].\end{aligned}$$

As explained earlier, the quantity $\phi(b) - \phi(a)$ represents the total angle through which the tangent vector $\mathbf{T}(t)$ turns as t varies from a to b. We can determine this angle just by looking at FIGURE 1. The two loops in the middle can be ignored, because the clockwise loop cancels the counterclockwise loop. The remaining part of the curve amounts to one-and-a-half turns clockwise. (Note that we finish our run going in the opposite direction from our starting direction.) Thus the total turning angle is -3π, and $L_2 - L_1 = \alpha(-3\pi)$. Taking $\alpha = 1$ yard, we conclude that I ran 3π yards farther than Rover. No wonder I was so tired!

In our calculation we assumed that $|\,\mathbf{r}_1'(t)\,| + \alpha\phi'(t) \geq 0$ for all t. Suppose this were violated, so that $|\,\mathbf{r}_1'(t)\,| + \alpha\phi'(t) < 0$ at some time t. Letting $R_1(t)$ be the signed radius of curvature of my path, and using equation (8), it would follow that $-\alpha < R_1(t) < 0$. This would mean that my path was turning clockwise with a radius of curvature $|\,R_1(t)\,| < \alpha$. I would be turning *towards* Rover so sharply that he could not compensate by slowing down. Instead he would have to do some additional running around (perhaps on a rather small scale) in order to remain in the ideal position beside me. Our simple formula for $L_2 - L_1$ would no longer apply. The reader might like to think about what happens if, for example, I run clockwise around a circle of radius less than one yard while Rover remains exactly one yard to my right.

We have been using real vector notation for our parametrizations, but complex notation has its advantages. The reader is invited to investigate how much simpler the computations become when my path is parametrized by a complex-valued function $\zeta_1(t)$. For instance, our unit normal vector is simply $ie^{i\phi(t)}$, where $\phi(t)$, defined as before, is an argument function for $\zeta_1'(t)$.

From a purely mathematical point of view, we have been comparing the arc lengths of two *parallel curves*. The mathematics we have presented is certainly not new, but there seems to be no single reference from which it can be easily extracted. Many of the ideas are to be found scattered through the first 25 pages of Klingenberg [**4**], while Exercise 6 on p. 47 of do Carmo [**2**] states a special case of our formula for $L_2 - L_1$. Many excellent sources are available [**5**, **6**] for anyone interested in delving further into differential geometry.

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2. Manfredo P. do Carmo, *Differential Geometry of Curves and Surfaces*, Prentice-Hall, Englewood Cliffs, NJ, 1976.
3. William Fulton, *Algebraic Topology: A First Course*, Springer, New York, 1995.
4. Wilhelm Klingenberg, *A Course in Differential Geometry*, Springer, New York, 1978.
5. Barrett O'Neill, *Elementary Differential Geometry*, 2nd ed., Academic Press, San Diego, 1997.
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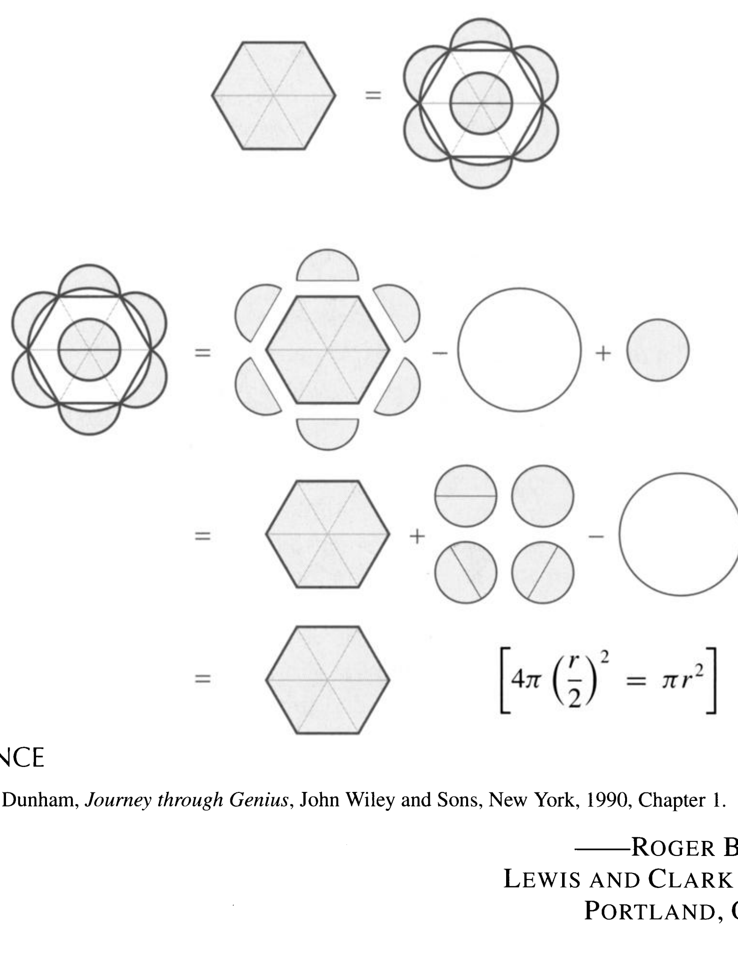
Proof Without Words: Lunes and the Regular Hexagon

THEOREM. If a regular hexagon is inscribed in a circle and six semicircles constructed on its sides, then the area of the hexagon equals the area of the six lunes plus the area of a circle whose diameter is equal in length to one of the sides of the hexagon. [Hippocrates of Chios, ca. 440 B.C.E]

Proof.



$$\left[4\pi\left(\frac{r}{2}\right)^2 = \pi r^2\right]$$

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