For more information on the lab which prompted this paper and which was first used in the spring of 1998 at the University of Minnesota, see http://www.macalester.edu/~oloughlin/math37/.

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Medical Tests and Convergence

STEPHEN FRIEDBERG
Illinois State University
Normal, IL 61790

In the syndicated column by Marilyn Vos Savant, a reader writes that she wants to be tested for a certain disease. However, she only wants to hear from her doctor if the news is good. Of course, having the doctor call her if and only if the result is good reveals bad news if the doctor does not call.

Marilyn suggests the following plan:

> Take the test, and have the laboratory send the results to the doctor in a sealed envelope. The doctor flips a coin. If a head appears, then he looks at the result. If she does not have the disease, he calls her. If she does have the disease, he does not call her. If a tail appears, then he does not look at the result, and he does not call her.
>
> This way, she only hears good news. If she hears nothing, she cannot conclude that the test result is bad.

Let us examine Marilyn's plan, using some basic real analysis and some results about difference equations. Two questions naturally come to mind.

Q1. Is the patient really more comfortable with hearing nothing than she was before she took the test?

Q2. Assuming that the patient is not called, what happens to the probability that she has the disease if the test result is sent to other doctors and a similar plan is used by each doctor?

The sample space may be described as consisting of the four outcomes, HD, HD', TD, and TD', where, for example, HD represents the outcome that the doctor obtains a head on the coin toss and the patient has the disease. The letter D' denotes the case that the patient does not have the disease. Let $p = \Pr(D)$ be the probability that she has the disease. Suppose C is the event she is called and C' the event that she is not called. We are interested in comparing $\Pr(D|C')$, the probability that she has the disease when she is not called, with p. Note that by Marilyn's plan, $\Pr(C'|H)$, the probability that she is not called given that the doctor flipped heads, is simply p. Using the fact that $D \subset C'$, we have

$$\Pr(D|C') = \frac{\Pr(D \cap C')}{\Pr(C')} = \frac{\Pr(D)}{\Pr(C')}.$$

Also

$$\Pr(C') = \Pr(C'|H)\Pr(H) + \Pr(C'|T)\Pr(T) = p(1/2) + (1)(1/2).$$

From the previous two equations, we conclude that

$$P(D|C') = \frac{p}{\frac{1}{2}p + \frac{1}{2}} = \frac{2p}{p+1} > p,$$

because $0 < p < 1$.

What does this mean? If she is not called, then she should be more worried under this plan than she was before she took the test. Therefore, the answer to our first question is "No," and for relatively rare diseases the "No" is even more resounding: the probability that she has the disease given that she is not called is about twice the probability that she has the disease at all.

Now suppose she is not called, and decides to seek the opinion of a second doctor, who will carry out the same scheme. For the second doctor, the probability that she has the disease is now $p_1 = 2p/(p+1)$. So, if she is not called a second time, the probability that she has the disease is $p_2 = 2p_1/(p_1+1)$. If her result is sent to n doctors, and she is not called by the n^{th} doctor, then the probability that she has the disease is

$$p_{n+1} = \frac{2p_n}{p_n + 1}. \qquad (*)$$

We use two very different methods for evaluating the limit of the sequence p_n.

Analysis Note from (*) that $\{p_n\}$ is an increasing sequence that is bounded above (by 2); hence it converges to some number c. Clearly, $c > 0$. Let $n \to \infty$ in (*), and obtain

$$c = 2c/(c+1).$$

It follows that $c = 1$. So we have $p_n \to 1$ as $n \to \infty$.

Difference equations We recognize that (*) is a first order nonlinear nonhomogeneous difference equation with initial condition $p_0 = p$. Solving this would be a good exercise for students. Elementary techniques (see Goldberg [**2**]) yield the solution:

$$p_{n+1} = \frac{2^{n+1}p}{p(2^{n+1}-1)+1} = \frac{p}{p(1-2^{1-n})+2^{1-n}} \to 1$$

as $n \to \infty$.

What does this mean for our patient hoping to avoid bad news? If she requests that this process be repeated many times because she has not heard any news, then the probability that she has the disease approaches 1 no matter how small the initial probability p is. This result not only conforms to our intuition, but our explicit formula for p_{n+1} allows us to check how quickly p_{n+1} converges to 1 for various values of p.

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Uniquely Determined Unknowns in Systems of Linear Equations

KENNETH HARDY
KENNETH S. WILLIAMS
Centre for Research in Algebra and Number Theory
School of Mathematics and Statistics
Carleton University
Ottawa, Ontario, Canada K1S 5B6
khardy@math.carleton.ca
williams@math.carleton.ca

BLAIR K. SPEARMAN
Department of Mathematics and Statistics
Okanagan University College
Kelowna, British Columbia, Canada V1V 1V7
bkspearm@okuc02.okanagan.bc.ca

Perhaps the reader has noticed that when solving a consistent system of linear equations (linear system) it can happen that some unknowns are uniquely determined, while others are not?

EXAMPLE. Consider the linear system

$$\begin{cases} 6x_1 + 12x_2 + x_3 + 6x_4 + x_5 = 7, \\ 5x_1 + 10x_2 + x_3 + 5x_4 + x_5 = 6, \\ 13x_1 + 26x_2 + 2x_3 + 13x_4 + 3x_5 = 18, \end{cases}$$

over the field $\mathbb{R}$ of real numbers. The solution set is

$$x_1 = 1 - 2s - t, \quad x_2 = s, \quad x_3 = -2, \quad x_4 = t, \quad x_5 = 3, \quad \text{where } s, t \in \mathbb{R},$$

and in this case x_3, x_5 are uniquely determined while x_1, x_2, x_4 can take infinitely many values.

This example suggests the following three questions.

QUESTION 1. *What is a necessary and sufficient condition for an unknown to be uniquely determined by a consistent linear system?*

QUESTION 2. *How many of the unknowns are uniquely determined by a linear system?*

QUESTION 3. *If an unknown is uniquely determined by a linear system, is there an explicit formula for it?*