Enter, Stage Center: The Early Drama of the Hyperbolic Functions

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In addition to the standard definitions of the hyperbolic functions (for instance, $\cosh x = (e^x + e^{-x})/2$), current calculus textbooks typically share two common features: a comment on the applicability of these functions to certain physical problems (for instance, the shape of a hanging cable known as the catenary) and a remark on the analogies that exist between properties of the hyperbolic functions and those of the trigonometric functions (for instance, the identities $\cosh^2 x - \sinh^2 x = 1$ and $\cos^2 x + \sin^2 x = 1$). Texts that offer historical sidebars are likely to credit development of the hyperbolic functions to the 18th-century mathematician Johann Lambert. Implicit in this treatment is the suggestion that Lambert and others were interested in the hyperbolic functions in order to solve problems such as predicting the shape of the catenary. Left hanging is the question of whether hyperbolic functions were developed in a deliberate effort to find functions with trig-like properties that were required by physical problems, or whether these trig-like properties were unintended and unforeseen by-products of the solutions to these physical problems. The drama of the early years of the hyperbolic functions is far richer than either of these plot lines would suggest.

Prologue: The catenary curve

What shape is assumed by a flexible inextensible cord hung freely from two fixed points? Those with an interest in the history of mathematics would guess (correctly) that this problem was first resolved in the late 17th century and involved the Bernoulli family in some way. The curve itself was first referred to as the "catenary" by Huygens in a 1690 letter to Leibniz, but was studied as early as the 15th century by da Vinci. Galileo mistakenly believed the curve would be a parabola [**8**]. In 1669, the German mathematician Joachim Jungius (1587–1657) disproved Galileo's claim, although his correction does not seem to have been widely known within 17th-century mathematical circles.

17th-century mathematicians focused their attention on the problem of the catenary when Jakob Bernoulli posed it as a challenge in a 1690 *Acta Eruditorum* paper in which he solved the isochrone problem of constructing the curve along which a body will fall in the same amount of time from any starting position. Issued at a time when the rivalry between Jakob and Johann Bernoulli was still friendly, this was one of the earliest challenge problems of the period. In June 1691, three independent solutions appeared in *Acta Eruditorum* [**1, 11, 16**]. The proof given by Christian Huygens employed geometrical arguments, while those offered by Gottfried Leibniz and Johann Bernoulli used the new differential calculus techniques of the day. In modern terminology, the crux of Bernoulli's proof was to show that the curve in question satisfies the differential equation $dy/dx = s/k$, where s represents the arc length from the vertex P to an arbitrary point Q on the curve and k is a constant depending on the weight per unit length of cord as in FIGURE 1.

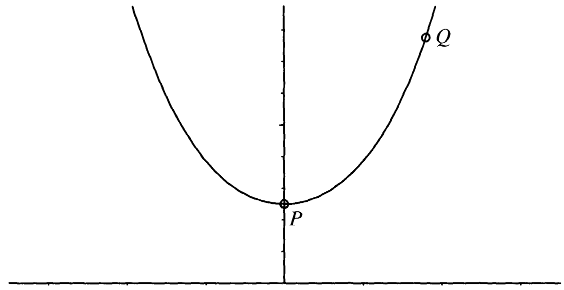

Figure 1 The catenary curve

Showing that $y = k\cosh(x/k)$ is a solution of this differential equation is an accessible problem for today's second-semester calculus student. 17th-century solutions of the problem differed from those of today's calculus students in a particularly notable way: *There was absolutely no mention of hyperbolic functions, or any other explicit function, in the solutions of 1691!* In these early days of calculus, curve constructions, and not explicit functions, were cast in the leading roles.

A suggestion of this earlier perspective can be heard in a letter dated September 19, 1718 sent by Johann Bernoulli to Pierre Réymond de Montmort (1678–1719):

> The efforts of my brother were without success; for my part, I was more fortunate, for I found the skill (I say it without boasting, why should I conceal the truth?) to solve it in full and to reduce it to the rectification of the parabola. It is true that it cost me study that robbed me of rest for an entire night. It was much for those days and for the slight age and practice I then had, but the next morning, filled with joy, I ran to my brother, who was still struggling miserably with this Gordian knot without getting anywhere, always thinking like Galileo that the catenary was a parabola. Stop! Stop! I say to him, don't torture yourself any more to try to prove the identity of the catenary with the parabola, since it is entirely false. The parabola indeed serves in the construction of the catenary, but the two curves are so different that one is algebraic, the other is transcendental ... (as quoted by Kline [**13**, p. 473]).

The term *rectification* in this passage refers to the problem of determining the arc length of a curve. The particular parabola used in Bernoulli's construction (given by $y = x^2/8 + 1$ in modern notation) was defined geometrically by Bernoulli as having "latus rectum quadruple the latus rectum of an equilateral hyperbola that shares the same vertex and axis" [**1**, pp. 274–275]. Bernoulli used the arc length of the segment of this parabola between the vertex $B = (0, 1)$ and the point $H = (\sqrt{8(y-1)}, y)$ to construct a segment GE such that the point E would lie on the catenary. In modern notation, the length of segment GE is the parabolic arc length BH, given by

$$\text{Arclength} = \sqrt{y^2-1} + \ln\left(y + \sqrt{y^2-1}\right),$$

while the catenary point E is given by

$$E = \left(-\ln\left(y + \sqrt{y^2-1}\right), y\right) = \left(x, \frac{e^x + e^{-x}}{2}\right).$$

The expression $\sqrt{y^2-1}$ in the arc length formula is the abscissa of the point $G(\sqrt{y^2-1}, y)$ on the equilateral hyperbola ($y^2 - x^2 = 1$) that played both the central role described above in defining the parabola necessary for the construction, as well as a supporting role in constructing the point E. Because a procedure for rectifying a parabola was known by this time, this reduction of the catenary problem to the rectification of a parabola provided a complete 17th-century solution to the catenary problem.

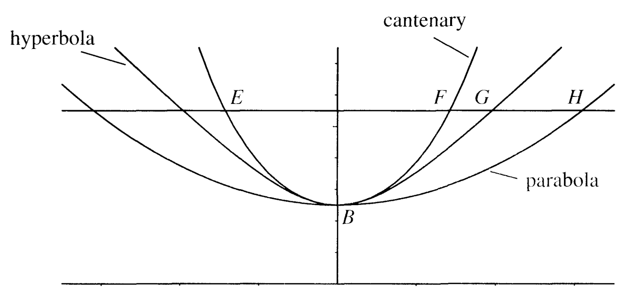

Figure 2 Bernoulli's construction of the catenary curve

Interestingly, another of the "first solvers" of the catenary problem, Christian Huygens, solved the rectification problem for the parabola as early as 1659. In fact, although the rectification problem had been declared by Descartes as beyond the capacity of the human mind [**4**, pp. 90–91], the problem of rectifying a curve C was known to be equivalent to the problem of finding the area under an associated curve C' by the time Huygens took up the parabola question.

A general procedure for determining the curve C' was provided by Hendrick van Heuraet (1634–1660) in a paper that appeared in van Schooten's 1659 Latin edition of Descartes' *La Geometrie*. (In modern notation, C' is defined by $L(t) = \int_a^t \sqrt{1 + (dy/dx)^2}\, dx$, where $y = f(x)$ defines the original curve C.) Huygens used this procedure to show that rectification of a parabola is equivalent to finding the area under a hyperbola. A solution of this latter problem in the study of curves—determining the area under a hyperbola—was first published by Gregory of St. Vincent in 1647 [**13**, p. 354]. Anton de Sarasa later recognized (in 1649) that St. Vincent's solution to this problem provided a method for computation of logarithmic values.

As impressive as these early "pre-calculus" calculus results were, by the time the catenary challenge was posed by Jakob Bernoulli in 1690, the rate at which the study of curves was advancing was truly astounding, thanks to the groundbreaking techniques that had since been developed by Isaac Newton (1642–1727) and Gottfried Leibniz (1646–1716). Relations between the Bernoulli brothers fared less well over the ensuing decades, as indicated by a later passage from Johann Bernoulli's 1718 letter to Montmort:

> But then you astonish me by concluding that my brother found a method of solving this problem.... I ask you, do you really think, if my brother had solved the problem in question, he would have been so obliging to me as not to appear among the solvers, just so as to cede me the glory of appearing alone on the stage in the quality of the first solver, along with Messrs. Huygens and Leibniz? (as quoted by Kline [**13**, p. 473])

Historical evidence supports Johann's claim that Jakob was not a "first solver" of the catenary problem. But in the year immediately following that first solution, Jakob Bernoulli and others solved several variations of this problem. Huygens, for example, used physical arguments to show that the curve is a parabola if the total load of cord and suspended weights is uniform per horizontal foot, while for the true catenary, the weight per foot along the cable is uniform. Both Bernoulli brothers worked on determining the shape assumed by a hanging cord of variable density, a hanging cord of constant thickness, and a hanging cord acted on at each point by a force directed to a fixed center. Johann Bernoulli also solved the converse problem: given the shape assumed by a flexible inelastic hanging cord, find the law of variation of density of the cord. Another nice result due to Jakob Bernoulli stated that, of all shapes that may be assumed by flexible inelastic hanging cord, the catenary has the lowest center of gravity.

A somewhat later appearance of the catenary curve was due to Leonhard Euler in his work on the calculus of variations. In his 1744 *Methodus Inveniendi Lineas Curvas Maximi Minimive Proprietate Gaudentes* [**5**], Euler showed that a catenary revolved about its axis (the catenoid) generates the only minimal surface of revolution. Calculating the surface area of this minimal surface is another straightforward exercise that can provide a nice historical introduction to the calculus of variations for second-semester calculus students. Kline [**13**, p. 579] comments that Euler himself did not make effective use of the full power of the calculus in the *Methodus*; derivatives were replaced by difference quotients, integrals by finite sums, and extensive use was made of geometric arguments. In tracing the story of the hyperbolic functions, this last point cannot be emphasized enough. From its earliest introduction in the 15th century through Euler's 1744 result on the catenoid, there is no connection made between analytic expressions involving the exponential function and the catenary curve. Indeed, prior to the development of 18th-century analytic techniques, no such connection could have been made. Calculus in the age of the Bernoullis was "the Calculus of Curves," and the catenary curve is just that—a *curve*. The hyperbolic functions did not, and could not, come into being until the full power of formal analysis had taken hold in the age of Euler.

Act I: The hyperbolic functions in Euler?

In seeking the first appearance of the hyperbolic functions as *functions*, one naturally looks to the works of Euler. In fact, the expressions $(e^x + e^{-x})/2$ and $(e^x - e^{-x})/2$ do make an appearance in Volume I of Euler's *Introductio in analysin infinitorum* (1745, 1748) [**6**]. Euler's interest in these expressions seems natural in view of the equations $\cos x = (e^{\sqrt{-1}x} + e^{-\sqrt{-1}x})/2$ and $\sqrt{-1}\sin x = (e^{\sqrt{-1}x} - e^{-\sqrt{-1}x})/2$ that he derived in this text. However, Euler's interest in what we call hyperbolic functions appears to have been limited to their role in deriving infinite product representations for the sine and cosine functions. Euler did not use the word *hyperbolic* in reference to the expressions $(e^x + e^{-x})/2$, $(e^x - e^{-x})/2$, nor did he provide any special notation or name for them. Nevertheless, his use of these expressions is a classic example of Eulerian analysis, included here as an illustration of 18th-century mathematics. An analysis of this derivation, either in its historical form or in modern translation, would be suitable for student projects in pre-calculus and calculus, or as part of a mathematics history course.

To better illustrate the style of Euler's analysis and the role played within it by the hyperbolic expressions, we employ his notation from the *Introductio* throughout this section. Although sufficiently like our own to make the work accessible to modern readers, there are interesting differences. For instance, Euler's use of periods in "$\sin . x$" and "$\cos . x$" suggests the notation still served as abbreviations for *sinus* and *cosinus*, rather than as symbolic function names. Like us, Euler and his contemporaries were intimately familiar with the infinite series representations for $\sin . x$ and $\cos . x$, but generally employed infinite series with less than the modern regard for rigor. Thus, as established in Section 123 of the *Introductio*, Euler could (and did) rewrite the expression $(e^x - e^{-x})$ as

$$e^x - e^{-x} = \left(1+\frac{x}{i}\right)^i - \left(1-\frac{x}{i}\right)^i = 2\left(\frac{x}{1} + \frac{x^3}{1\cdot 2\cdot 3} + \frac{x^5}{1\cdot 2\cdot 3\cdot 4\cdot 5} + \text{etc.}\right),$$

where i represented an infinitely large quantity (and *not* the square root of -1, denoted throughout the *Introductio* as $\sqrt{-1}$). Other results used by Euler are also familiarly unfamiliar to us, most notably the fact that $a^n - z^n$ has factors of the form $aa - 2az\cos.2k\pi/n + zz$, as established in Section 151 of the *Introductio*.

Euler's development of infinite product representations for $\sin.x$ and $\cos.x$ in the *Introductio* begins in Section 156 by setting $n = i$, $a = 1 + x/i$, and $z = 1 - x/i$ in the expression $a^n - z^n$ (where, again, i is infinite, so, for example, $a^n = (1 + x/i)^i = e^x$). After some algebra, the result of Section 151 cited above allowed Euler to conclude that $e^x - e^{-x}$ has factors of the form $2 - 2xx/ii - 2(1 - xx/ii)\cos.2k\pi/n$. Substituting $\cos.2k\pi/n = 1 - (2kk/ii)\pi\pi$ (the first two terms of the infinite series representation for cosine) into this latter expression and doing a bit more algebra, Euler obtained the equation

$$2 + \frac{2xx}{ii} - 2\left(1 - \frac{xx}{ii}\right)\cos.2k\frac{\pi}{n} = \frac{4xx}{ii} + \left(\frac{4kk}{ii}\right)\pi\pi - \frac{4kk\pi\pi xx}{i^4}.$$

Ergo (to quote Euler), $e^x - e^{-x}$ has factors of the form $1 + xx/(kk\pi\pi) - xx/ii$. Since i is an infinitely large quantity, Euler's arithmetic of infinite and infinitesimal numbers allowed the last term to drop out. (Tuckey and McKenzie give a thorough discussion of these ideas [**17**].) The end result of these calculations, as presented in Section 156, thereby became

$$\begin{aligned}\frac{e^x - e^{-x}}{2} &= x\left(1+\frac{xx}{\pi\pi}\right)\left(1+\frac{xx}{4\pi\pi}\right)\left(1+\frac{xx}{9\pi\pi}\right)\left(1+\frac{xx}{16\pi\pi}\right)\text{etc.}\\ &= 1 + \frac{xx}{1\cdot 2\cdot 3} + \frac{x^4}{1\cdot 2\cdot 3\cdot 4\cdot 5} + \frac{x^6}{1\cdot 2\cdot 3\cdot 4\cdot 5\cdot 6\cdot 7} + \text{etc.} \qquad (1)\end{aligned}$$

A similar calculation (Section 157) derived the analogous series for $(e^x + e^{-x})/2$.

In Section 158, Euler employed these latter two results in the following manner. Recalling the well-known fact (which he derived in Section 134) that

$$\frac{e^{z\sqrt{-1}} - e^{z\sqrt{-1}}}{2\sqrt{-1}} = \sin.z = z - \frac{z^3}{1\cdot 2\cdot 3} + \frac{z^5}{1\cdot 2\cdot 3\cdot 4\cdot 5} - \text{etc.},$$

Euler let $x = z\sqrt{-1}$ in equation (1) above to get

$$\begin{aligned}\sin.z &= z\left(1-\frac{zz}{\pi\pi}\right)\left(1-\frac{zz}{4\pi\pi}\right)\left(1-\frac{zz}{9\pi\pi}\right)\left(1-\frac{zz}{16\pi\pi}\right)\text{etc.}\\ &= z\left(1-\frac{z}{\pi}\right)\left(1+\frac{z}{\pi}\right)\left(1-\frac{z}{2\pi}\right)\left(1-+\frac{z}{2\pi}\right)\left(1-\frac{z}{3\pi}\right)\text{etc.}\end{aligned}$$

The same substitution, applied to the series with even terms, yielded the now-familiar product representation for $\cos.z$.

Here we arrive at Euler's apparent goal: the derivation of these lovely infinite product representations for the sine and cosine. Although the expressions $(e^x + e^{-x})/2$ and

$(e^x - e^{-x})/2$ played a role in obtaining these results, it was a supporting role, with the arrival of the hyperbolic functions on center stage yet to come.

Act II, Scene I: Lambert's first introduction of hyperbolic functions

Best remembered today for his proof of the irrationality of π, and considered a forerunner in the development of noneuclidean geometries, Johann Heinrich Lambert was born in Mülhasen, Alsace on August 26, 1728. The Lambert family had moved to Mülhasen from Lorraine as Calvinist refugees in 1635. His father and grandfather were both tailors. Because of the family's impoverished circumstances (he was one of seven children), Lambert left school at age 12 to assist the family financially. Working first in his father's tailor shop and later as a clerk and private secretary, Lambert accepted a post as a private tutor in 1748 in the home of Reichsgraf Peter von Salis. As such, he gained access to a good library that he used for self-improvement until he resigned his post in 1759. Lambert led a largely peripatetic life over the next five years. He was first proposed as a member of the Prussian Academy of Sciences in Berlin in 1761. In January 1764, he was welcomed by the Swiss community of scholars, including Euler, in residence in Berlin. According to Scriba [**21**], Lambert's appointment to the Academy was delayed due to "his strange appearance and behavior." Eventually, he received the patronage of Frederick the Great (who at first described him as "the greatest blockhead") and obtained a salaried position as a member of the physical sciences section of the Academy on January 10, 1765. He remained in this position, regularly presenting papers to each of its divisions, until his death in 1777 at the age of 49.

Lambert was a prolific writer, presenting over 150 papers to the Berlin Academy in addition to other published and unpublished books and papers written in German, French, and Latin. These included works on philosophy, logic, semantics, instrument design, land surveying, and cartography, as well as mathematics, physics, and astronomy. His interests appeared at times to shift almost randomly from one topic to another, and often fell outside the mainstream of 18th-century science and mathematics. We leave it to the reader to decide whether his development of the hyperbolic functions is a case in point, or an exception to this tendency.

Lambert first treated hyperbolic trigonometric functions in a paper presented to the Berlin Academy of Science in 1761 that quickly became famous: *Mémoire sur quelques propriétés remarquables des quantités transcendantes circulaires et logarithmiqes* [**14**]. Rather than its consideration of hyperbolic functions, this paper was (and is) celebrated for giving the first proof of the irrationality of π. Lambert established this long-awaited result using continued fractions representations to show that z and $\tan z$ cannot both be rational; thus, since $\tan(\pi/4)$ is rational, π can not be.

Instead of concluding the paper at this rather climatic point, Lambert turned his attention in the last third of the paper to a comparison of the "*transcendantes circulaire*" [$\sin v$, $\cos v$,] with their analogues, the "*quantités transcendantes logarithmiques*" [$(e^v + e^{-v})/2$, $(e^v - e^{-v})/2$]. Beginning in Section 73, he first noted that the transcendental logarithmic quantities can be obtained from the transcendental circular quantities by taking all the signs in

$$\sin v = v - \frac{1}{2\cdot 3}v^3 + \frac{1}{2\cdot 3\cdot 4\cdot 5}v^5 - \frac{1}{2\cdot 3\cdot 4\cdot 5\cdot 6\cdot 7}v^7 + \text{etc.}$$

to be positive, thereby obtaining

$$\frac{e^v - e^{-v}}{2} = v + \frac{1}{2\cdot 3}v^3 + \frac{1}{2\cdot 3\cdot 4\cdot 5}v^5 + \frac{1}{2\cdot 3\cdot 4\cdot 5\cdot 6\cdot 7}v^7 + \text{etc.},$$

and similarly for the cosine series. He then derived continued fraction representations (in Section 74) for the expressions $(e^v - e^{-v})/2$, $(e^v + e^{-v})/2$, and $(e^v - e^{-v})/(e^v + e^{-v})$, and noted that these continued fraction representations can be used to show that v and e^v cannot both be rational. The fact that none of its powers or roots are rational prompted Lambert to speculate that e satisfied *no* algebraic equation with rational coefficients, and hence is *transcendental*. Charles Hermite (1822–1901) finally proved this fact in 1873. (Ferdinand Lindemann (1852–1939) established the transcendence of π in 1882.)

Although Lambert did not introduce special notation for his "*quantités transcendantes logarithmiques*" in this paper, he did go on to develop the analogy between these functions and the circular trigonometric functions that he said "should exist" because

> ... the expressions $e^u + e^{-u}$, $e^u - e^{-u}$, by substituting $u = v\sqrt{-1}$, give the circular quantities $e^{v\sqrt{-1}} + e^{-v\sqrt{-1}} = 2\cos v$, $e^{v\sqrt{-1}} - e^{-v\sqrt{-1}} = 2\sin v \cdot \sqrt{-1}$.

Lambert was especially interested in developing this "affinity" as far as possible without introducing imaginary quantities. To do this he introduced (in Section 75) a parameterization of an "equilateral hyperbola" ($x^2 - y^2 = 1$) to define the hyperbolic functions in a manner directly analogous to the definition of trigonometric functions by means of a unit circle ($x^2 + y^2 = 1$). Lambert's parameter is twice the area of the hyperbolic sector shown in FIGURE 3. Lambert used the letter M to denote a typical point on the hyperbola, with coordinates (ξ, η).

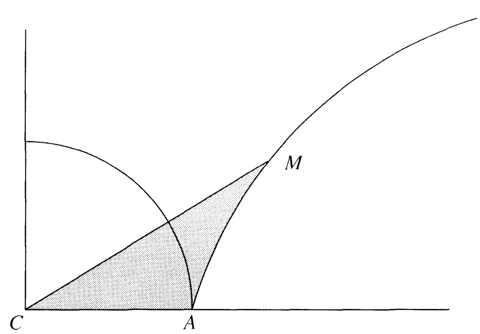

Figure 3 The parameter u represents twice the area of the shaded sector MCA

In Lambert's own diagram (FIGURE 4), the circle and the hyperbola are drawn together. The letter C marks the common center of the circle and the hyperbola, CA is the radius of the circle, CF the asymptote of the hyperbola, and AB the tangent line common to the circle and the hyperbola. The typical point on the hyperbola corresponds to a point N on the circle, with coordinates (x, y). Lowercase letters m and n mark nearby points on the hyperbola and circle, for use in differential computations.

Denoting the angle MCA by ϕ, Lambert listed several differential properties for quantities defined within this diagram, using a two-columned table intended to display the similarities between the "*logarithmiques*" and "*circulaires*" functions. The first seven lines of this table, reproduced below, defined the necessary variables and stated basic algebraic and trigonometric relations between them. Note especially the third line of this table, where $u/2$ (which Lambert denoted as $u : 2$) is defined to be the area of the hyperbolic "segment" $AMCA$.

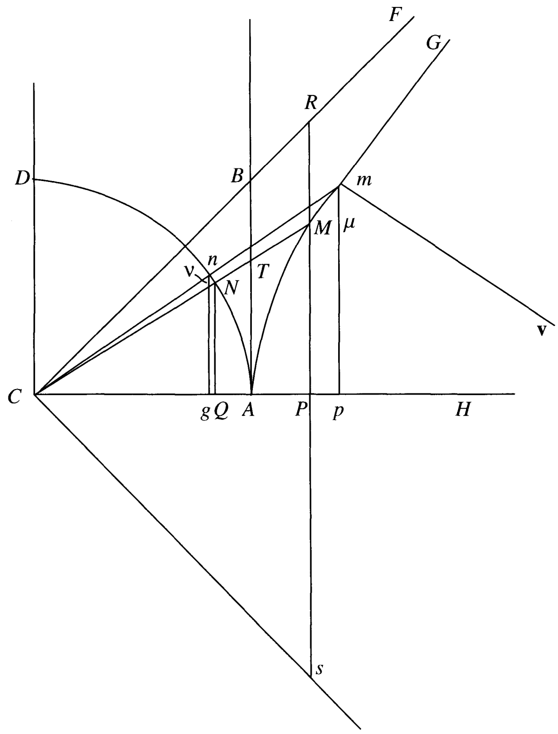

Figure 4 Diagram from Lambert's 1761 *Mémoire*

	pour l'hyperbole	*pour le cercle*
l'abscisse	$CP = \xi \ldots$	$\ldots CQ = x$
l'ordonné	$PM = \eta \ldots$	$\ldots QN = y$
le segment	$AMCA = u : 2 \ldots$	$\ldots ANCA = v : 2$
et il sera		
	$\tang \phi = \frac{\eta}{\xi} \ldots$	$\ldots \tang \phi = \frac{y}{x}$
	$1 + \eta\eta = \xi\xi = \eta\eta \cot \phi^2 \ldots$	$\ldots 1 - yy = xx = yy \cot \phi^2$
	$\xi\xi - 1 = \eta\eta = \xi\xi \tang \phi^2 \ldots$	$\ldots 1 - xx = yy = xx \tang \phi^2$
	$CM^2 = \xi^2 + \eta^2$	$CN^2 = x^2 + y^2$
	$= \xi^2(1 + \tang \phi^2) = \frac{1+\tang \phi^2}{1-\tang \phi^2}$	$= x^2(1 + \tang \phi^2) = \frac{1+\tang \phi^2}{1+\tang \phi^2} = 1$

Using these relations, it is a straightforward exercise to derive expressions for the differentials $d\xi$, $d\eta$, dx, and dy (as a step toward finding infinite series expressions for ξ and η). For example, given $\xi\xi - 1 = \eta\eta = \xi\xi \tang \phi^2$ (tang would be tan in modern notation), it follows that $\xi = 1/\sqrt{1 - \tang \phi^2}$. Lambert noted this fact, along with the differential $d\xi = \tang \phi \, d \tang \phi/(1 - \tang \phi^2)^{3:2}$ obtained from it, later in the table.

To see how differential expressions for du and dv might be obtained, note that u is defined to be twice the area of the hyperbolic sector $AMCA$. The differential du thus represents twice the area of the hyperbolic sector MCm. This differential sector can be approximated by the area of a circular sector of radius CM and angle $d\phi$; that is, $du = 2[CM^2 d\phi/2]$. Substituting $CM^2 = (1 + \tang \phi^2)/(1 - \tang \phi^2)$ from the table above then yields $du = d\phi \cdot CM^2 = d\phi \cdot (1 + \tang \phi^2)/(1 - \tang \phi^2)$, where $d\phi \cdot (1 + \tang \phi^2) = d(\tang \phi)$. Thus, $du = d \tang \phi/(1 - \tang \phi^2)$. Although Lambert omitted the details of these derivations, his table summarized them as shown below.

pour l'hyperbole	*pour le cercle*
$+\,du = d\phi \cdot \left(\frac{1+\text{tang}\,\phi^2}{1-\text{tang}\,\phi^2}\right)$	$dv = d\phi = \frac{d\,\text{tang}\,\phi}{1+\text{tang}\,\phi^2}$
$= \frac{d\,\text{tang}\,\phi}{1-\text{tang}\,\phi^2}$	
$+\,d\xi = \frac{\text{tang}\,\phi d\,\text{tang}\,\phi}{(1-\text{tang}\,\phi^2)^{3:2}}$	$-\,dx = \frac{\text{tang}\,\phi d\,\text{tang}\,\phi}{(1+\text{tang}\,\phi^2)^{3:2}}$
$+\,d\eta = \frac{d\,\text{tang}\,\phi}{(1-\text{tang}\,\phi^2)^{3:2}}$	$+\,dy = \frac{d\,\text{tang}\,\phi}{(1+\text{tang}\,\phi^2)^{3:2}}$
$\vdots$	$\vdots$
$+\,d\xi : du = \eta \ldots$	$\ldots - dx : dv = y$
$+\,d\eta : du = \xi \ldots$	$\ldots + dy : dv = x$
$+\,d\xi = d\eta \cdot \text{tang}\,\phi \ldots$	$\ldots - dx : dy = \text{tang}\,\phi$

Using the relations $+\,d\xi : du = \eta$, $+\,d\eta : du = \xi$ from this table, along with standard techniques of the era for determining the coefficients of infinite series, Lambert then proved (Section 77) that the following relations hold:

$$\eta = u + \frac{1}{2\cdot 3}u^3 + \frac{1}{2\cdot 3\cdot 4\cdot 5}u^5 + \frac{1}{2\cdot 3\cdot 4\cdot 5\cdot 6\cdot 7}u^7 + \text{etc.}$$

$$\xi = u + \frac{1}{2}u^2 + \frac{1}{2\cdot 3\cdot 4}u^4 + \frac{1}{2\cdot 3\cdot 4\cdot 5\cdot 6}u^6 + \text{etc.},$$

where we recall that ξ is the abscissa of a point on the hyperbola, η is the ordinate of that same point, and u represents twice the area of the hyperbolic segment determined by that point. But these are exactly the infinite series for $(e^u - e^{-u})/2$ and $(e^u + e^{-u})/2$ with which Lambert began his discussion of the "*quantités transcendantes logarithmiques*."

Lambert was thus able to conclude (Section 78) that $\xi = (e^u - e^u)/2$ and $\eta = (e^u - e^{-u})/2$ are, respectively, the abscissa and ordinate of a point on the hyperbola for which u represents twice the area of the hyperbolic segment determined by that point.

A derivation of this result employing integration, as outlined in some modern calculus texts, is another nice problem for students. Contrary to the suggestion of some texts, it is this parameterization of the hyperbola by the hyperbolic sine and cosine, and the analogous parameterization of the circle by the circular sine and cosine, that seems to have motivated Lambert and others eventually to provide the hyperbolic functions with trig-like names—not the similarity of their analytic identities. This is not to say that the similarities between the circular identities and the hyperbolic identities were without merit in Lambert's eyes—we shall see that Lambert and others exploited these similarities for various purposes. But Lambert's immediate interest in his 1761 paper lay elsewhere, as we shall examine more closely in the following section.

Interlude: Giving credit where credit is due

As Lambert himself remarked at several points in his 1761 *Mémoire*, he was especially interested in developing the analogy between the two classes of functions (circular versus hyperbolic) as far as possible without the use of imaginary quantities, and it is the geometric representation (that is, the parameterization) that provides him a means to this end. Lambert ascribed his own interest in this theme to the work of another 18th-century mathematician whose name is less well known, Monsieur le Chevalier François Daviet de Foncenex.

As a student at the Royal Artillery School of Turin, de Foncenex studied mathematics under a young Lagrange. As recounted by Delambre, the friendships Lagrange formed with de Foncenex and other students led to the formation of the Royal Academy of Science of Turin [7]. A major goal of the society was the publication of mathematical and scientific papers in their *Miscellanea Taurinensia*, or *Mélanges de Turin*. Both Lagrange and de Foncenex published several papers in early volumes of the *Miscellanea*, with de Foncenex crediting Lagrange for much of the inspiration behind his own work. Delambre argued that Lagrange provided de Foncenex with far more than inspiration, and it is true that de Foncenex's analytic style is strongly reminiscent of Lagrange. It is also true that de Foncenex did not live up to the mathematical promise demonstrated in his early work, although he was perhaps sidetracked from a mathematical career after being named head of the navy by the King of Sardinia as a result of his early successes in the *Miscellanea*.

In his earliest paper, *Reflexions sur les Quantités Imaginaire* [7], de Foncenex focused his attention on "the nature of imaginary roots" within the debate concerning logarithms of negative quantities. In particular, de Foncenex wished to reconcile Euler's "incontestable calculations" proving that negative numbers have imaginary logarithms with an argument from Bernoulli that opposed this conclusion on grounds involving the continuity of the hyperbola (whose quadrature defines logarithms) at infinity. The analysis that de Foncenex developed of this problem led him to consider the relation between the circle and the equilateral hyperbola—exactly the same analogy pursued by Lambert.

In his 1761 *Mémoire*, Lambert fully credited de Foncenex with having shown how the affinity between the circular trigonometric functions and the hyperbolic trigonometric functions can be "seen in a very simple and direct fashion by comparing the circle and the equilateral hyperbola with the same center and same diameter." De Foncenex himself went no further in exploring "this affinity" than to conclude that, since $\sqrt{x^2 - r^2} = \sqrt{-1}\sqrt{r^2 - x^2}$, "the circular sectors and hyperbolic [sectors] that correspond to the same abscissa are always in the ratio of 1 to $\sqrt{-1}$." It is this use of an imaginary ratio to pass from the circle to the hyperbola Lambert seemed intent on avoiding.

Lambert returned to this theme one final time in Section 88 of the *Mémoire*. In another classic example of 18th-century analysis, Lambert first remarked that "one can easily find by using the differential formulas of Section 75," that

$$v = \operatorname{tang}\phi - \frac{1}{3}\operatorname{tang}\phi^3 + \frac{1}{5}\operatorname{tang}\phi^5 - \frac{1}{7}\operatorname{tang}\phi^7 + \text{etc.}$$

$$\operatorname{tang}\phi = u - \frac{1}{3}u^3 + \frac{2}{15}u^5 - \frac{17}{315}u^7 + \text{etc.}$$

"By substituting the value of the second of these series into the first ... and reciprocally" (but again with details omitted), Lambert obtained the following two series:

$$v = u - \frac{2}{3}u^3 + \frac{2}{3}u^5 - \frac{244}{315}u^7 + \text{etc.} \tag{2}$$

$$u = v + \frac{2}{3}v^3 + \frac{2}{3}v^5 + \frac{244}{315}v^7 + \text{etc.}$$

where (switching from previous usage) u equals twice the area of the circular sector and v equals twice the area of the hyperbolic sector. Finally, Lambert obtained the sought-after relation by noting that substitution of $u = v\sqrt{-1}$ into series (2) will yield $v = u\sqrt{-1}$.

In (semi)-modern notation, we can represent Lambert's results as $\tanh(v\sqrt{-1}) = \tan(u\sqrt{-1})$ and $\tanh(u) = \tan(v)$. Having thus established that imaginary hyperbolic sectors correspond to imaginary circular sectors, and similarly for real sectors, Lambert closed his 1761 *Mémoire*. The next scene examines how he later pursued a new plot line suggested by this analogy: the use of hyperbolic functions to replace circular functions in the solution of certain problems.

Act II, Scene II: The reappearance of hyperbolic functions in Lambert

Lambert returned to the development of his "transcendental logarithmic functions" and their similarities to circular trigonometric functions in his 1768 paper *Observations trigonometriques* [**15**]. In this treatment, a typical point on the hyperbola is called q. Letting ϕ denote the angle qCQ in FIGURE 5, Lambert first remarked that $\text{tang}\,\phi = MN/MC = qp/pC$. Because $MN/MC = \sin\phi/\cos\phi$ and $qp/pC = \sin\text{hyp}\,\phi/\cos\text{hyp}\,\phi$, one has the option of using either the circular tangent function or the hyperbolic tangent functions for the purpose of analyzing triangle qCP. *Note that the notation and terminology used here are Lambert's own!* Lambert himself commented that, in view of the analogous parameterizations that are possible for the circle and the hyperbola, there is "nothing repugnant to the original meaning" of the terms "sine" and "cosine" in the use of the terms "hyperbolic sine" and "hyperbolic cosine" to denote the abscissa and ordinate of the hyperbola. Although Lambert's notation for these functions differed from our current convention, the hyperbolic functions had now become fully-fledged players in their own right, complete with names and notation suggestive of their relation to the circular trigonometric functions.

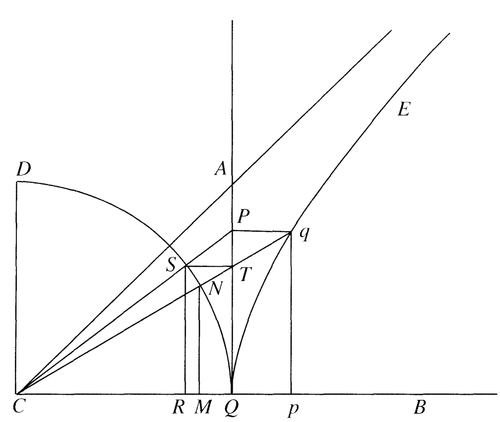

Figure 5 Diagram from Lambert's *Observations trigonometriques* [**15**]

The development of the hyperbolic functions in this paper included an extensive list of sum, difference, and multi-angle identities that are, as Lambert remarked, easily derived from the formulas $\sin\text{hyp}\, v = (e^v + e^{-v})/2$, $\cos\text{hyp}\, v = (e^v - e^{-v})/2$. Of greater importance to Lambert's immediate purpose was the table of values he constructed for certain functions of "the transcendental angle ω." In particular, the transcendental angle ω, defined as angle PCQ in FIGURE 5 and related to the common angle ϕ via the relation $\sin\omega = \text{tang}\,\phi = \text{tang}\,\text{hyp}\,\phi$, served Lambert as a means to pass from circular functions to hyperbolic functions. (The transcendental angle associated with ϕ is also known as the *hyperbolic amplitude of* ϕ after Hoüel and the *longitude* after Guderman.) For values of ω ranging from 1° to 90° in increments of 1 degree, Lambert's table included values of the hyperbolic sector, the hyperbolic sine and its logarithm, the hyperbolic cosine and its logarithm, as well as the tangent of the corresponding common angle and its logarithm. By replacing circular functions by hyperbolic functions, Lambert used these functions to simplify the computations required to determine the angle measures and the side lengths of certain triangles.

The triangles that Lambert was interested in analyzing with the aid of the hyperbolic functions arise from problems in astronomy in which one of the celestial bodies is below the horizon. It has since been noted that such problems can be solved using formulae from spherical trigonometry with arcs that are pure imaginaries. This is an intriguing observation since elsewhere (in his work on noneuclidean geometry), Lambert speculated on the idea that a sphere of imaginary radius might reflect the geometry of "the acute angle hypothesis." The acute angle hypothesis is one of three possibilities for the two (remaining) similar angles α, β of a quadrilateral assumed to have two right angles and two congruent sides: (1) angles α, β are right; (2) angles α, β are obtuse; and (3) angles α, β are acute. Girolamo Saccheri (1667–1733) introduced this quadrilateral in his *Euclides ab omi naevo vindiactus* of 1733 as an element of his efforts to prove Euclid's Fifth Postulate by contradiction. Both Saccheri and Lambert believed they could dispense with the obtuse angle hypothesis. Lambert's speculation about the acute angle hypothesis was the result of his inability to reject the acute angle hypothesis.

It is worth emphasizing, however, that Lambert himself never put an imaginary radius into the formulae of spherical trigonometry in any of his published works. The triangles he treated are real triangles with real-valued arcs and real-valued sides. As noted by historian Jeremy Gray [**9**, pp. 156–158], the ability to articulate clearly the notion of "geometry on a sphere of imaginary radius" was not yet within the grasp of mathematicians in the age of Euler. Gray argues convincingly that the development of analysis by Euler, Lambert, and other 18th-century mathematicians was, nevertheless, critical for the 19th-century breakthroughs in the study of noneuclidean geometry. By providing a language flexible enough to discuss geometry in terms other than those set forth by Euclid, analytic formulae allowed for a reformulation of the problem and the recognition that a new geometry for space was possible. Although rarely mentioned in today's calculus texts, the explicit connection eventually made by Beltrami in his 1868 paper, linking the hyperbolic functions to the noneuclidean geometry of an imaginary sphere, is yet another intriguing use for hyperbolic functions that is surely as tantalizing as the oft-cited catenary curve.

Flashback: Hyperbolic functions in Riccati Although Lambert's primary reason for considering hyperbolic functions in 1768 was to simplify calculations involved in solving triangles, Lambert clearly realized that there was no need to define new functions for this purpose; tables of logarithms of appropriate trigonometric values could instead be used to serve the same end. But, he argued, this was only one possible use for the hyperbolic trigonometric functions in mathematics. The only example he cited in this regard was the simplification of solution methods for equations. Lambert did not elaborate on this idea beyond noting that the equation $0 = x^2 - 2a\cos\omega \cdot x + a^2$ is equivalent to the equation $0 = x^2 - 2a\text{coshyp}\psi \cdot x + a^2$ for an appropriately defined angle ψ. He did, however, cite an investigation of this idea that had already appeared in the work of another 18th-century mathematician: Vincenzo de Riccati.

Vincenzo de Riccati was born on January 11, 1707, the second son of Jacopo Riccati for whom the Riccati equation in differential equations is named. Riccati (the son) received his early education at home and from the Jesuits. He entered the Jesuit order in 1726 and taught or studied in various locations, including Piacenza, Padua, Parma, and Rome. In 1739, Riccati moved to Bologna, where he taught mathematics in the College of San Francesco Saverio until Pope Clement XIV suppressed the Society of Jesus in 1773. Riccati then returned to his family home in Treviso, where he died on January 17, 1775.

Riccati first treated hyperbolic functions in his two-volume *Opuscula ad res physicas et mathematicas pertinentium* (1757–1762) **[19]**. In this work, Riccati employed a hyperbola to define functions that he referred to as "sinus hyperbolico" and "cosinus hyperbolico," doing so in a manner analogous to the use of a circle to define the functions "sinus circulare" and "cosinus circulare." Taking u to be the quantity given by twice the area of the sector ACF divided by the length of the segment CA (whether in the circle or the hyperbola of FIGURE 6), Ricatti defined the sine and cosine of the quantity u to be the segments GF and CG of the appropriate diagram. Although Ricatti did not explicitly assume either a unit circle or an equilaterial hyperbola, his definitions are equivalent to that of Lambert (and our own) in that case. In *Opusculum* IV of Volume I, Ricatti derived several identities of his hyperbolic sine and cosine, applying these to the problem of determining roots of certain equations, especially cubics. Riccati also determined the series representations for the *sinus* and *cosinus hyperbolicos*. These latter results, which appeared in *Opusculum* VI in volume I, were earlier communicated by Riccati to Josepho Suzzio in a letter dated 1752.

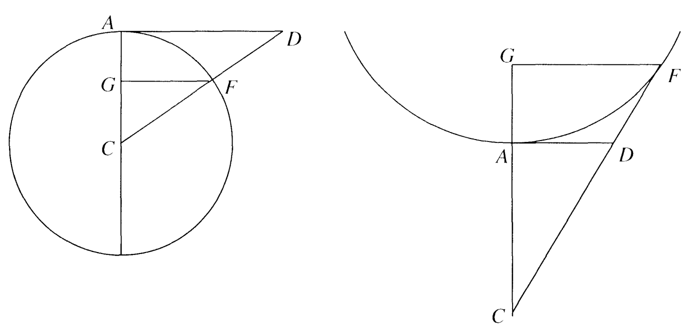

Figure 6 Diagrams rendered from Riccati's *Opuscula*

In Riccati's *Institutiones analyticae* (1765–1767) **[20]**, written collaboratively with Girolamo Saldini, he further developed the theory of the hyperbolic functions, including the standard addition formulas and other identities for hyperbolic functions, their derivatives and their relation to the exponential function (already implicit in his *Opuscula*).

Reprise: Giving credit where credit is due While some of the ideas in Riccati's *Institutiones* of 1765–1767 also appeared in Lambert's 1761 *Mémoire*, this author knows of no evidence to suggest that Riccati was building on Lambert's work. The publication dates of his earlier work suggest that Riccati was familiar with the analogy between the circular and the hyperbolic functions some time earlier than Lambert came across the idea, and certainly no later. Conversely, even though Riccati's earliest work was published several years before Lambert's 1761 *Mémoire*, it appears that Lambert was unfamiliar with Riccati's work at that time. Certainly, the motivations of the two for introducing the hyperbolic functions appear to have been quite different. Furthermore, Lambert appears to have been scrupulous in giving credit to colleagues when drawing on their work, as in the case of de Foncenex. In fact, Lambert credited Ricatti with developing the terminology "hyperbolic sine" and "hyperbolic cosine" when he used these names for the first time in his 1768 *Observations trigonometriques*. It thus appears that it was only these new names—and perhaps the idea of using these functions to solve equations—that Lambert took from Riccati's work, finding them to be suitable nomenclature for mathematical characters whom he had already developed within a story line of his own creation.

Despite the apparent independence of their work, the fact remains that Riccati did have priority in publication. Why then is Lambert's name almost universally mentioned in this context, with Riccati receiving little or no mention? Histories of mathematics written in the 19th and early 20th centuries suggest this tendency to overlook Riccati's work is a relatively recent phenomenon. Von Braunmühl [**22**, pp. 133–134], for example, has the following to say in his 1903 history of trigonometry:

> In fact, Gregory St. Vincent, David Gregory and Craig through the quadrature of the equilateral hyperbola, erected the foundations [for the hyperbolic functions], even if unaware of the fact, Newton touched on the parallels between the circle and the equilateral hyperbola, and de Moivre seemed to have some understanding that, by substituting the real for the imaginary, the role of the circle is replaced by the equilateral hyperbola. Using geometric considerations, Vincenzo Riccati (1707–1775) was the first to found the theory of hyperbolic functions, as was recognized by Lambert himself. *(Author's translation.)*

Although the amount of recognition that Lambert afforded Riccati may be overestimated here, it is interesting that von Braunmühl then proceeded to discuss Lambert's work on hyperbolic functions in detail, with no further mention of Riccati, remarking that:

> This [hyperbolic function] theory is only of interest to us in so far as it came into use in the treatment of trigonometric problems, as was first opened up by Lambert. *(Author's translation.)*

It would thus appear that the motivation Lambert assigned to the hyperbolic functions was more central to mathematical interests as they evolved thereafter, even though his interests often fell outside the mainstream of his own century. The fact that Lambert's mathematical works, especially those on noneuclidean geometry, were studied by his immediate mathematical successors offers support for this idea, as does the wider availability of Lambert's works today. Besides being more widely available, Lambert's work is written in notation—and languages!—that are more familiar to today's scholars than that of Riccati. This alone makes it easier to tell Lambert's story in more detail, just as we have done here.

Epilogue

And what of the physical applications for which the hyperbolic functions are so useful? Although neither Lambert nor Riccati appear to have studied these connections, they were known by the late 19th century, as evidenced by the publication of hyperbolic function tables and manuals for engineers in that period. Yet even as late as 1849, we hear Augustus De Morgan [**3**, p. 66], declare:

> The system of trigonometry, from the moment that $\sqrt{-1}$ is introduced, always presents an incomplete and one-sided appearance, unless the student have in his mind for comparison *(though it is rarely or never wanted for what is called use)*, another system [hyperbolic trigonometry] in which the there-called sines and cosines are real algebraic quantities. (Emphasis added.)

While De Morgan's perspective offers yet another intriguing reason to study hyperbolic trigonometry, usefulness in solving problems (mathematical or physical) did not appear to concern him. This delay between the development of the mathematical machinery and its application to physical problems serves as a gentle reminder that the physical applications we sometimes cite as the raison d'être for a mathematical idea may only become visible with hindsight. Yet even Riccati's and Lambert's own uses for hyperbolic trigonometry went unacknowledged by De Morgan—an even stronger reminder of how quickly mathematics changed in the 19th century, and how greatly today's mathematics classroom might be enriched by remembering the mathematics of the age of Euler.

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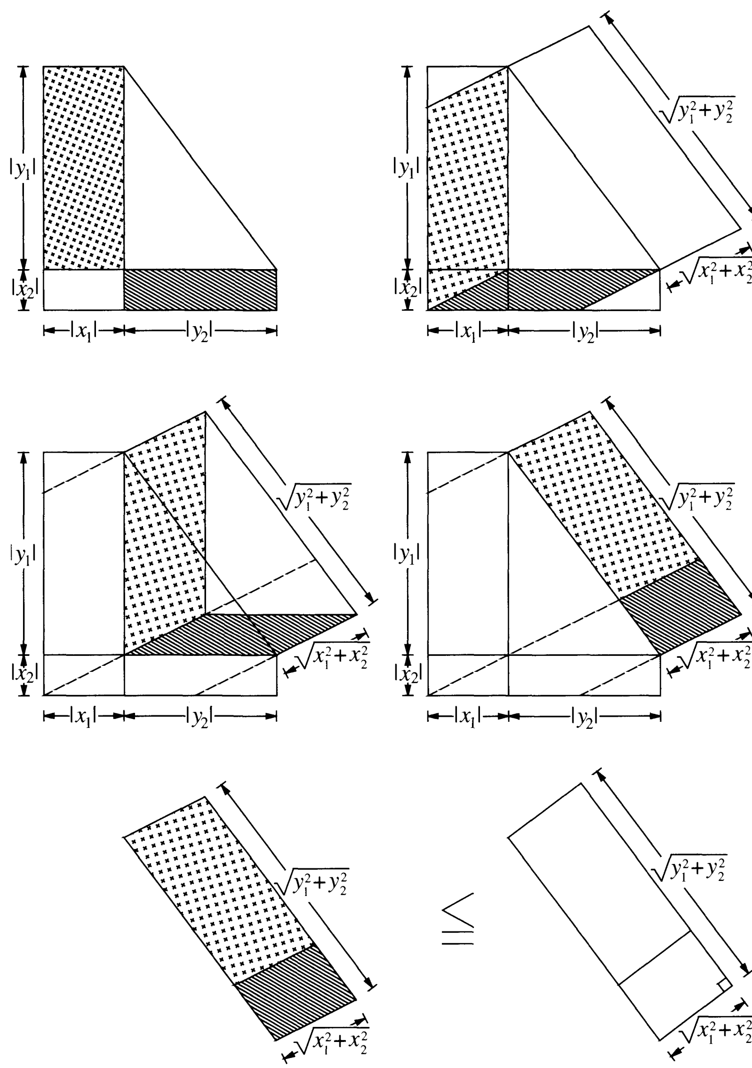
Proof Without Words: Cauchy-Schwarz Inequality

$$|x_1y_1 + x_2y_2 \leq| \, |x_1| \cdot |y_1| + |x_2| \cdot |y_2| \leq \sqrt{x_1^2 + x_2^2} \cdot \sqrt{y_1^2 + y_2^2}$$



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