$S_7(15)$ mod 105:	7	14	28	49	56	77	91	98
$U(15)$ mod 15:	7	14	13	4	11	2	1	8

The subgroup $G = \{1, 11\}$ of $U(15)$ gives the subgroup $\Phi(G) = \{91, 56\}$ of $S_7(15)$. The subgroup $G = \{1, 2, 4, 8\}$ gives $\Phi(G) = \{91, 77, 98, 49\}$; and $G = \{1, 4, 11, 14\}$ gives $\Phi(G) = \{91, 49, 56, 14\}$. All of these $\Phi(G)$ are groups under multiplication modulo 105. Who would ever have guessed that! By using subgroups, the fact that these examples were obtained from $U(15)$ is no longer obvious.

Another interesting challenge might be to produce sets that are groups under multiplication modulo 2005. Since $2005 = 5 \cdot 401$, we can take $k = 5$, $n = 401$ and consider $S_5(401)$ under multiplication modulo 2005. The inverse of 5 in $U(401)$ is $k' = 321$, hence the identity in $S_5(141)$ will be $E = 5 \cdot 321 = 1605$. And any subgroup of $U(401)$ will produce a subgroup of $S_5(141)$. Consider $G = \{39, 318, 372, 72, 1\}$, the cyclic subgroup of $U(401)$ generated by 39. Multiply each element by 1605 then reduce modulo 2005, to obtain a group under multiplication modulo 2005: $\Phi(G) = \{440, 1120, 1575, 1275, 1605\}$. This is, of course, a cyclic group generated by $\Phi(39) = 440$.

Robin McLean [**2**] takes a different approach to these results. He also points out that one can pick any positive composite integer and produce a group that has this integer as its identity element. This can be done by observing $E \equiv 1 \bmod (E - 1)$. For example, if we want $E = 123$, consider $n = 122$. Since $E = 123 = 3 \cdot 41$, we can take $k = 3$, with inverse $k' = 41$ in $U(122)$. In $S_3(122)$ the identity element (mod $3 \cdot 122$) will be $E = kk' = 123$. Of course $S_3(122)$ has 60 elements, just like $U(122)$, but remember that any of its subgroups will also have 123 as identity element. For example: $\{123, 135, 291\}$ is a group under multiplication mod 366. Have fun producing your own examples!

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The St. Basil's Cake Problem

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Perhaps the most popular cake in the Greek world during the Christmas period is not any well-known Christmas cake, but rather a cake called the St. Basil's cake. (St. Basil is commemorated by the Greek Orthodox Church on the first of January.) The cake is prepared using simple ingredients like flour, eggs, and orange juice, but also contains a coin wrapped in aluminum foil. When the cake is taken out of the oven, nobody knows where the coin is. With the arrival of the new year, this circular cake is cut (with a knife) into sectors of the circle. Each member of the family takes a sector and starts slowly and carefully eating his or her piece. The one who finds the coin, according to tradition, is considered to be the luckiest of the new year. However, sometimes, while the cake is being cut into sectors, the knife hits the coin. This event happens quite often, and therefore it might be of interest to calculate its probability. Let's call this problem the *St. Basil's cake problem.*

Mathematical formulation First of all, observe that under the assumption that the coin is parallel to the base of the cake, the problem is equivalent to the following: A circle in the plane is divided into sectors. A coin is dropped at random on a circle, in a certain sense that we will clarify. We would like to calculate the probability that the coin is contained entirely in any one of the sectors.

The problem reminds us of the well known *Buffon's needle problem*, and in fact it might be considered as an extension or a variant of that. Let us very briefly describe the problem of the needle of Buffon. Consider a needle of length l that is dropped at random on a set of equidistant parallel lines that are d units apart, with $l < d$. What is the probability of an intersection? Uspenski [**2**] proved that this probability is $2l/\pi d$. His proof is a little bit complicated; a simpler proof is presented in Solomon [**1**], who also gives further details and various extensions and generalizations.

Solution Let us suppose that we have a circle of radius R^*, which is divided into $2n$ sectors with central angle π/n, and a coin with radius R_c. We must obviously assume that $R_c < R^*$. The coin is dropped at random on the circle by selecting a radius at random and locating the coin randomly along that radius.

The parameters that define the coin's position involve a vector from the center of the cake to the center of the coin. The length of this vector is r, and ϕ is the angle it makes with the closest cut of the circle. Then we can safely assume that r and ϕ are independent random variables. If the distance r is greater than $R^* - R_c$, then the coin would be partially outside of the circle. Thus, $r < R \equiv R^* - R_c$. The way we located the coin leads us to assume that r follows the uniform distribution on the interval $(0, R)$ and that ϕ follows the uniform distribution on the interval $(0, \pi/2n)$.

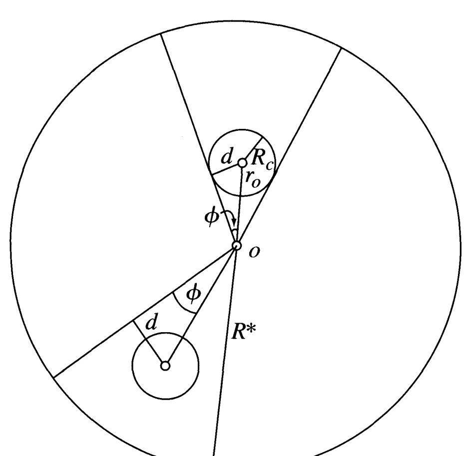

Figure 1 Locating the coin

Let d denote the distance of the center of the coin from the closest radius. As it can be seen in FIGURE 1, the coin is contained entirely in a sector if $r > r_0$, where r_0 satisfies the equation

$$\sin\left(\frac{\pi}{2n}\right) = \frac{R_c}{r_0}, \quad \text{that is,} \quad r_0 = \frac{R_c}{\sin(\pi/2n)}$$

and $d > R_c$, with the condition on d being equivalent to $r\sin\phi > R_c$, that is, $\phi > \arcsin(R_c/r)$.

Remark. We must assume that $R > r_0$ or equivalently, that $R\sin(\pi/2n) > R_c$, for otherwise the probability of an intersection is 1.

The probability that the coin is contained entirely in a sector is

$$\begin{aligned} P\big(r > r_0, \phi > \arcsin(R_c/r)\big) &= \int_{r_0}^{R}\int_{\arcsin(R_c/r)}^{\pi/2n} \frac{1}{R}\cdot\frac{1}{\pi/2n}\,d\phi\,dr \\ &= \int_{r_0}^{R}\left(\frac{1}{R} - \frac{2n}{\pi R}\arcsin(R_c/r)\right)dr \\ &= \frac{r}{R}\bigg|_{r_0}^{R} - \int_{r_0}^{R}\frac{2n}{\pi R}\arcsin(R_c/r)\,dr \\ &= 1 - \frac{R_c}{R\sin(\pi/2n)} - \frac{2n}{\pi R}\int_{r_0}^{R}\arcsin(R_c/r)\,dr \\ &= 1 - \frac{R_c}{R\sin(\pi/2n)} - \frac{2n}{\pi R}I, \end{aligned} \tag{1}$$

where we have used I as a shorthand for the integral on the line above. We will calculate I separately. For notational simplicity let $\Lambda = R_c/R$. Using the transformation $R_c/r = \sin u$, the integral I becomes

$$\begin{aligned} I &= \int_{\pi/2n}^{\arcsin(\Lambda)} -R_c u\cos(u)\big(\sin(u)\big)^{-2}\,du = \int_{\pi/2n}^{\arcsin(\Lambda)} R_c u\,d\Big(\big(\sin(u)\big)^{-1}\Big) \\ &= R_c\,u\big(\sin(u)\big)^{-1}\Big|_{\pi/2n}^{\arcsin(\Lambda)} - \int_{\pi/2n}^{\arcsin(\Lambda)} R_c\big(\sin(u)\big)^{-1}du \\ &= R_c\,u\big(\sin(u)\big)^{-1}\Big|_{\pi/2n}^{\arcsin(\Lambda)} - R_c\log\big|\tan(u/2)\big|\Big|_{\pi/2n}^{\arcsin(\Lambda)} \\ &= R_c\arcsin(\Lambda)\Lambda^{-1} - R_c(\pi/2n)\big(\sin(\pi/2n)\big)^{-1} \\ &\quad -R_c\log\big|\tan\big(\arcsin(\Lambda)/2\big)\big| + R_c\log\big|\tan(\pi/4n)\big|. \end{aligned}$$

Substituting the value of I into (1) and collecting terms together we have that

$$\begin{aligned} &P\big(r > r_0, \phi > \arcsin(R_c/r)\big) \\ &= 1 - \frac{2n}{\pi}\arcsin(\Lambda) + \frac{2n}{\pi}\Lambda\log\big|\tan\big(\arcsin(\Lambda)/2\big)\big| - \frac{2n}{\pi}\Lambda\log\big|\tan(\pi/4n)\big| \\ &= 1 - \frac{2n}{\pi}\left(\arcsin(\Lambda) + \Lambda\log\left|\frac{\tan(\pi/4n)}{\tan(\arcsin(\Lambda)/2)}\right|\right). \end{aligned}$$

Thus the probability, p, that one of the radial cuts hits the coin is

$$p = \frac{2n}{\pi}\left(\arcsin(\Lambda) + \Lambda\log\left|\frac{\tan(\pi/4n)}{\tan(\arcsin(\Lambda)/2)}\right|\right). \tag{2}$$

Numerical application Using expression (2) from above, we can examine p for specific values of R^* and R_c, and for various values of n. For instance assume that we have the realistic scenario that the radius of the coin is $R_c = 1$ cm and that of the cake is $R^* = 16$ cm. TABLE 1 gives the value of p for selected values of n. Surprisingly enough, we observe that even when n is equal to 2, which means that the cake is cut into four equal parts, the probability that the knife hits the coin is very high (close to 30%). As expected, p is increasing as n increases and for $n = 23$, p is just below 1. For $n \geq 24$, the probability of an intersection is clearly 1 since for those values it holds true that $R \sin(\pi/2n) \leq R_c$.

TABLE 1: Probability of intersection for selected values of n.

n	2	4	8	12	16	20	22	23
p	0.2987	0.4730	0.7073	0.8535	0.9422	0.9881	0.9978	0.9997

Remark In this note, we consider only the case where the coin is dropped at random on a circle in the plane. However, various extensions of that problem can be formulated. For example, in $\mathbb{R}^3$ one might consider the situation where the coin takes any position in an object like a cylinder, or a sphere, not necessarily in a uniform manner.

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Replacement Costs: the Inefficiencies of Sampling with Replacement

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When we deal five cards in poker, we do not deal the same card twice. The cards dealt are all distinct (unless the deck is rigged). This is typical of most sampling problems, where samples are chosen *without replacement*. This means that once chosen, an object is not eligible to be selected again.

However, there are interesting practical problems involving sampling *with replacement*. For example, one might need to evaluate the performance of a computer program intended to generate 5-digit random numbers. If the numbers are truly random, then each digit is equally likely to be any of the values 0–9 and the digits are independent of one another. Thus if the program is working properly, we can consider each 5-digit number to be a sample of size five chosen with replacement from the popula-