Power Series Expansions for Trigonometric Functions via Solutions to Initial Value Problems

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1. Introduction The coefficients in the power series expansion of $\sec(\phi)$, valid in an interval $|\phi| < \pi/2$, involve certain integers that are generally referred to as Euler's numbers. These numbers, denoted by E_p, with p a nonnegative integer, may be considered as known through a recursion relation. Tables, such as those that appear in [1], give the values of some of these numbers. There is, however, no simple formula that gives E_p explicitly; E_{60}, for example, is a positive integer containing 71 digits. As a consequence, power series expansions for functions involving products of secant functions become rather complicated and explicit formulas for the coefficients in the expansions are rather difficult to obtain. For example, a power series expansion for $\sec^2(\phi)$ could be obtained from the Cauchy product of the series expansion of $\sec(\phi)$ with itself. The coefficient of ϕ^{2n} could be easily determined in terms of the Euler numbers. However, expansions for higher powers of $\sec(\phi)$, such as $\sec^3(\phi)$ or $\sec^4(\phi)$, have more complicated formulas for the coefficients, though the Cauchy product will, in principle, yield these expressions. Similar remarks may be made concerning the power series expansion of $\tan(\phi)$ in an interval $|\phi| < \pi/2$. The coefficients that appear in this case involve the Bernoulli numbers, B_p.

In this note an alternative to the Cauchy product method is given for the determination of power series that represent products of certain trigonometric functions. The method involves repeated differentiation of a differential equation and is introduced in Section 3, which follows a summary in Section 2 of certain elementary results from calculus concerning the power series expansions of $\sec(\phi)$ and $\sec^2(\phi)$. The method discussed in Section 3 is applicable to a limited class of functions. It is, however, also a method that may lead more readily than Cauchy products to power series representations of these functions. Moreover, it is applicable as a method of obtaining power series solutions to certain initial value problems.

2. Power series expansions of $\sec(\phi)$ and $\sec^2(\phi)$ The power series expansion of $\sec(\phi)$, valid for $|\phi| < \pi/2$, is given by

$$\sec(\phi) = \sum_{n=0}^{\infty} \frac{(-1)^n E_{2n}}{(2n)!} \phi^{2n}. \tag{2.1}$$

The numbers E_{2n} are the Euler numbers and their values may be found using the recursion relation

$$\sum_{k=0}^{n} \binom{2n}{2k} E_{2n-2k} = 0, \tag{2.2}$$

where $E_0 = 1$.

The formula (2.2) is obtained from a Cauchy product of the series for $\sec(\phi)$ and $\cos(\phi)$. Thus,

$$\cos(\phi)\sec(\phi) = \sum_{n=0}^{\infty}\left(\sum_{k=0}^{n}\frac{E_{2n-2k}}{(2k)!(2n-2k)!}\right)(-1)^n\phi^{2n} = 1. \tag{2.3}$$

Since $E_0 = 1$, formula (2.2) then gives the following values for the next six Euler numbers:

$$\begin{aligned} E_2 &= -1\\ E_4 &= 5\\ E_6 &= -61\\ E_8 &= 1385\\ E_{10} &= -50521\\ E_{12} &= 2{,}702{,}765. \end{aligned} \tag{2.4}$$

A power series expansion for $\sec^2(\phi)$ is then obtained by taking the Cauchy product of the series for $\sec(\phi)$ with itself. The result is

$$\sec^2(\phi) = 1 + \sum_{n=1}^{\infty}\left(\sum_{k=0}^{n}\binom{2n}{2k}E_{2k}E_{2n-2k}\right)\frac{(-1)^n}{(2n)!}\phi^{2n}. \tag{2.5}$$

We note that the odd order derivatives of $\sec(\phi)$ and $\sec^2(\phi)$ vanish when $\phi = 0$. The even order derivatives are given from (2.1) and (2.5) by

$$(\sec(\phi))^{(2n)}\big|_{\phi=0} = (-1)^n E_{2n} \tag{2.6a}$$

$$\left(\sec^2(\phi)\right)^{(2n)}\big|_{\phi=0} = (-1)^n\sum_{k=0}^{n}\binom{2n}{2k}E_{2k}E_{2n-2k}. \tag{2.6b}$$

Hence, for example,

$$(\sec(\phi))^{(8)}\big|_{\phi=0} = E_8 = 1385,$$

while

$$\begin{aligned}\left(\sec^2(\phi)\right)^{(6)}\big|_{\phi=0} &= -\left\{\binom{6}{0}E_0E_6 + \binom{6}{2}E_2E_4 + \binom{6}{4}E_4E_2 + \binom{6}{6}E_6E_0\right\}\\ &= -\{-61 + 15(-5) + 15(-5) - 61\} = 272.\end{aligned}$$

3. Power series expansions A power series expansion for $\sec^4(\phi)$ could be obtained by a Cauchy product of the series for $\sec^2(\phi)$ with itself. An alternative method, which we present here, relies on certain nice features of the secant function. First, let us observe that if $u(\phi) = \sec(\phi)$, then u is a solution of the initial value problem

$$\begin{aligned}(u')^2 + u^2 &= \sec^4(\phi)\\ u(0) &= 1.\end{aligned} \tag{3.1}$$

If we denote evaluations of the derivatives of u at $\phi = 0$ by the subscript 0, then repeated differentiation of the equation (3.1) followed by evaluations at $\phi = 0$ yields

$$u_0^{(1)}\left[u_0^{(2)} + u_0^{(0)}\right] = 2\sec^4(\phi)\tan\phi\big|_{\phi=0} = \left[\frac{\sec^4(\phi)}{2}\right]^{(1)}_{\phi=0}$$

$$u_0^{(2)}\left[u_0^{(2)}+u_0^{(0)}\right]+u_0^{(1)}\left[u_0^{(3)}+u_0^{(1)}\right]=\left[\frac{\sec^4(\phi)}{2}\right]_{\phi=0}^{(2)}$$

$$u_0^{(3)}\left[u_0^{(2)}+u_0^{(0)}\right]+2u_0^{(2)}\left[u_0^{(3)}+u_0^{(1)}\right]+u_0^{(1)}\left[u_0^{(4)}+u_0^{(2)}\right]=\left[\frac{\sec^4(\phi)}{2}\right]_{\phi=0}^{(3)}.$$

Only even order derivatives can yield nonzero results since $u_0^{(2k-1)}=0$ for every positive integer k. Hence a general formula for $[\sec^4(\phi)/2]_{\phi=0}^{(2n)}$ can be obtained, and we find

$$\sum_{k=0}^{n-1}\binom{2n-1}{2k}u_0^{(2n-2k)}\left[u_0^{(2k+2)}+u_0^{(2k)}\right]=\left[\frac{\sec^4(\phi)}{2}\right]_{\phi=0}^{(2n)}. \tag{3.2}$$

Since $u_0^{(2p)}=[\sec(\phi)]_{\phi=0}^{(2p)}=(-1)^pE_{2p}$, we must then have

$$\sum_{k=0}^{n-1}\binom{2n-1}{2k}(-1)^{n+1}E_{2n-2k}(E_{2k+2}-E_{2k})=\left[\frac{\sec^4(\phi)}{2}\right]_{\phi=0}^{(2n)}. \tag{3.3}$$

Thus we have obtained a power series expansion for $\sec^4(\phi)$ given by

$$\sec^4(\phi)=1+2\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{(2n)!}\left(\sum_{k=0}^{n-1}\binom{2n-1}{2k}E_{2n-2k}(E_{2k+2}-E_{2k})\right)\phi^{2n}. \tag{3.4}$$

The first few terms in the series expansion are

$$\sec^4(\phi)=1+2\phi^2+\frac{7}{3}\phi^4+\frac{94}{45}\phi^6+\frac{502}{315}\phi^8+\cdots. \tag{3.5}$$

A similar procedure could be used to obtain a power series for $\sec^6(\phi)$. In this case we observe that $u(\phi)=\sec^2(\phi)$ is a solution to the initial value problem

$$(u')^2+4u^2=4\sec^6(\phi) \tag{3.6}$$
$$u(0)=1.$$

Repeated differentiation of the equation (3.6) followed by evaluation at $\phi=0$ yields

$$\sum_{k=0}^{n-1}\binom{2n-1}{2k}u_0^{(2n-2k)}\left[u_0^{(2k+2)}+4u_0^{(2k)}\right]=\left[2\sec^6(\phi)\right]_{\phi=0}^{(2n)}, \tag{3.7}$$

where

$$u_0^{(2p)}=\left[\sec^2(\phi)\right]_{\phi=0}^{(2p)}=(-1)^p\sum_{k=0}^{p}\binom{2p}{2k}E_{2k}E_{2p-2k}. \tag{3.8}$$

Hence we obtain an expansion for $\sec^6(\phi)$ given by

$$\sec^6(\phi)=1+\frac{1}{2}\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{(2n)!}\left(\sum_{k=0}^{n-1}\binom{2n-1}{2k}\left(\sum_{j=0}^{n-k}\binom{2n-2k}{2j}E_{2j}E_{2n-2k-2j}\right)\Lambda\right)\phi^{2n}, \tag{3.9}$$

where

$$\Lambda = \sum_{l=0}^{k+1} \binom{2k+2}{2l} E_{2l} E_{2k-2l+2} - 4 \sum_{l=0}^{k} \binom{2k}{2l} E_{2l} E_{2k-2l}. \tag{3.10}$$

The first few terms in the expansion (3.9) are given by

$$\sec^6(\phi) = 1 + 3\phi^2 + 5\phi^4 + \frac{92}{15}\phi^6 + \cdots . \tag{3.11}$$

The success of this method hinged in part on the fact that we chose a particular differential equation whose known solution, an even function, could be repeatedly differentiated and whose derivatives at 0 were known. It is possible to devise other examples for which this method could be applied. The fact that the solutions of (3.1) and (3.6) are even functions is not crucial in the procedure of repeated differentiation of the differential equation. For example, the initial value problem

$$\begin{aligned} u'' - 2u &= 2\tan^3\phi, \ 0 < \phi < \pi/2 \\ u(0) &= 0 \\ u'(0) &= 1 \end{aligned} \tag{3.12}$$

has as its unique solution $u(\phi) = \tan\phi$, an odd function. The procedure of repeated differentiation of the differential equation may be applied in this case to obtain a series expansion for $\tan^3(\phi)$, since the derivatives of $\tan(\phi)$ at $\phi = 0$ are known. Since

$$\begin{aligned} \tan(\phi) &= \sum_{n=1}^{\infty} \frac{(-1)^k 2^{2k}(2^{2k}-1)}{(2k)!} B_{2k}\phi^{2k-1} \\ &= \phi + \frac{1}{3}\phi^3 + \frac{2}{15}\phi^5 + \frac{17}{315}\phi^7 + \frac{62}{2835}\phi^9 + \cdots, \end{aligned} \tag{3.13}$$

where the B_{2k} are Bernoulli numbers, the first few of which are

$$\begin{aligned} B_2 &= \frac{1}{6} & B_8 &= -\frac{1}{30} \\ B_4 &= -\frac{1}{30} & B_{10} &= \frac{5}{66} \\ B_6 &= \frac{1}{42} \end{aligned} \tag{3.14}$$

we are then led to equations analogous to (3.2) and (3.3). The results are

$$u_0^{(2k+1)} - 2u_0^{(2k-1)} = \left[2\tan^3(\phi)\right]_{\phi=0}^{(2k-1)} \tag{3.15}$$

and, hence,

$$\left[2\tan^3(\phi)\right]_{\phi=0}^{(2k-1)} = (-1)^k 2^{2k+1}\left\{\frac{2(2^{2k+2}-1)B_{2k+2}}{(2k+2)} + \frac{(2^{2k}-1)B_{2k}}{(2k)}\right\}. \tag{3.16}$$

The series expansion for $\tan^3(\phi)$ is then given by

$$\tan^3(\phi) = \sum_{k=2}^{\infty} \frac{(-1)^k 2^{2k}}{(2k-1)!}\left[\frac{2(2^{2k+2}-1)B_{2k+2}}{(2k+2)} + \frac{(2^{2k}-1)B_{2k}}{(2k)}\right]\phi^{2k-1} \tag{3.17}$$

$$= \phi^3 + \phi^5 + \frac{11}{15}\phi^6 + \frac{88}{189}\phi^9 + \cdots . \tag{3.18}$$

Still another example, from which an expansion for $\sec^3(\phi)$ may be found, is provided by the initial value problem

$$\begin{aligned} u'' + u &= \left[2\sec^3\phi\right] \\ u(0) &= 1 \\ u'(0) &= 0. \end{aligned} \tag{3.19}$$

It is easily verified that $u(\phi) = \sec(\phi)$ is a solution. The procedure of repeatedly differentiating the differential equation then results in the expansion

$$\begin{aligned} \sec^3(\phi) &= \sum_{k=0}^{\infty} \frac{(-1)^{k-1}}{2(2k)!}[E_{2k+2} - E_{2k}]\phi^{2k} \\ &= 1 + \frac{3}{2}\phi^2 + \frac{11}{8}\phi^4 + \cdots, \end{aligned} \tag{3.20}$$

valid on the interval $|\phi| < \pi/2$.

The expansions obtained in (3.4), (3.17), and (3.20) are simpler in form than those that would have been obtained by taking Cauchy products. On the other hand, the process by which these series were obtained is somewhat special in that the solution to an initial value problem must be known, along with the values of the derivatives of the solution at 0. Thus the initial value problem has to be carefully chosen. We note also that the process of repeated differentiation of a differential equation may sometimes be used to generate coefficients for series solutions. The reader interested in this application should see **[3]**, for example.

REFERENCES

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2. K. Knopp, *Theory and Application of Infinite Series*, Hafner Publishing Co., New York, 1971.
3. W. E. Boyce and R. C. DiPrima, *Elementary Differential Equations*, 3rd edition, John Wiley and Sons, Inc., New York, 1983, p. 166.

"It always seems to me absurd to speak of a complete proof, or of a theorem being rigorously demonstrated. An incomplete proof is no proof, and a mathematical truth not rigorously demonstrated is not demonstrated at all."

—*Sylvester's Collected Works*, Vol. 2, p. 200