determinant due to Mansion [3], although he did not state his method as a matrix factorization. See Muir's history [4] for more details.

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A Laplace Transform Technique for Evaluating Infinite Series

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In an article in this MAGAZINE, Efthimiou [2] shows how the Laplace transform can be used as a tool for evaluating infinite series. We review his method and illustrate a more general application of the technique.

Efthimiou's technique Efthimiou [2] finds closed-form expressions for series of the form

$$\sum_{n=1}^{\infty} \frac{1}{(n+a)(n+b)}, \quad \text{where} \quad a, b \notin \{-1, -2, -3, \ldots\}. \tag{1}$$

He applies the same methods to sum series of the form $\sum_{n=1}^{\infty} Q(n)/P(n)$, where P and Q are polynomials with $\deg(P) - \deg(Q) = 2$ and P factors completely into linear factors with no roots in $\{1, 2, 3, \ldots\}$.

Efthimiou's technique applies when $\sum_{n=1}^{\infty} u_n$ is an infinite series whose summand u_n can be realized as a Laplace transform integral $u_n = \int_0^{\infty} e^{-nx} f(x)\, dx$. In such a case, what appeared to be a sum of numbers is now written as a sum of integrals. This may not seem like progress, but interchanging the order of summation and integration (with proper justification of course!) yields a sum that we can evaluate easily, namely, a geometric series. Here are the steps:

$$\sum_{n=1}^{\infty} u_n = \sum_{n=1}^{\infty} \int_0^{\infty} e^{-nx} f(x)\,dx$$

$$\stackrel{?}{=} \int_0^{\infty} f(x) \sum_{n=1}^{\infty} e^{-nx}\,dx = \int_0^{\infty} f(x) \left(\frac{e^{-x}}{1-e^{-x}} \right) dx.$$

If this integral is easy to evaluate, we will have our sum.

To illustrate this technique, consider the series (1) where $a \neq b$; without loss of generality, we can assume that $b > a > -1$. The nth term can be written (use partial fractions) as the Laplace transform integral

$$\frac{1}{(n+a)(n+b)} = \int_0^{\infty} e^{-nx} \left(\frac{e^{-ax} - e^{-bx}}{b-a} \right) dx,$$

giving a starting point for the steps above.

We must justify changing the order of summation and integration. Since the integrands are all nonnegative for $0 < x < \infty$, we can apply the monotone convergence theorem (see, for instance, Folland [**3**, p. 49]) to switch the sum and the integral and obtain

$$\begin{aligned}
\sum_{n=1}^{\infty} \frac{1}{(n+a)(n+b)} &= \sum_{n=1}^{\infty} \int_0^{\infty} e^{-nx} \left(\frac{e^{-ax} - e^{-bx}}{b-a} \right) dx \\
&\stackrel{M.C.T.}{=} \int_0^{\infty} \left(\frac{e^{-ax} - e^{-bx}}{b-a} \right) \sum_{n=1}^{\infty} e^{-nx}\,dx \\
&= \frac{1}{b-a} \int_0^{\infty} (e^{-ax} - e^{-bx}) \left(\frac{e^{-x}}{1-e^{-x}} \right) dx \\
&= \frac{1}{b-a} \int_0^1 \frac{u^a - u^b}{1-u}\,du. \qquad (2)
\end{aligned}$$

As a first example, take $a = 0$ and $b = 1/2$; then (2) yields

$$\sum_{n=1}^{\infty} \frac{1}{n(n+\frac{1}{2})} = 2 \int_0^1 \frac{1-u^{\frac{1}{2}}}{1-u}\,du = 2 \int_0^1 \frac{1}{1+u^{\frac{1}{2}}}\,du = 4(1-\ln 2).$$

For another example, take $a = 0$ and b a positive integer; then (2) yields the popular telescoping series identity

$$\begin{aligned}
\sum_{n=1}^{\infty} \frac{1}{n(n+b)} &= \frac{1}{b} \int_0^1 \frac{1-u^b}{1-u}\,du = \frac{1}{b} \int_0^1 \left(1 + u + u^2 + \cdots + u^{b-1}\right) du \\
&= \frac{1}{b} \left(1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{b} \right).
\end{aligned}$$

We refer the reader to Efthimiou's paper [**2**] to see a rich application of this technique to other series.

A more general technique Efthimiou's technique can be generalized to series of the form $\sum_{n=1}^{\infty} u_n v_n$ where it is convenient to write only v_n as a Laplace transform integral. Again, the series can be written as a sum of integrals, but this time there is a factor of u_n before each integral. If the order of summation and integration can be interchanged (again, with proper justification!), we will need to find an explicit sum for the series $h(x) = \sum_{n=1}^{\infty} u_n e^{-nx}$. If all this works out, we will have

$$\sum_{n=1}^{\infty} u_n v_n = \sum_{n=1}^{\infty} u_n \int_0^{\infty} e^{-nx} f(x)\,dx$$
$$\stackrel{?}{=} \int_0^{\infty} f(x) \left(\sum_{n=1}^{\infty} u_n e^{-nx} \right) dx = \int_0^{\infty} f(x)h(x)\,dx.$$

For example, consider

$$\sum_{n=1}^{\infty} \frac{r^n}{an+b}, \quad \text{where} \quad r \in [-1, 1), \quad a > 0, \quad \text{and} \quad b \geq 0. \tag{3}$$

The trick will be to recognize $1/(an+b)$ as being equal to the Laplace transform integral $= \int_0^{\infty} e^{-nx}(e^{-b/ax})/a\,dx$.

As long as $r \neq -1$, the partial sums of $\sum_{n=1}^{\infty} r^n e^{-nx}(e^{-b/ax})/a$ are dominated above by

$$\sum_{n=1}^{\infty} \left| r^n e^{-nx} \left(\frac{1}{a} e^{-\frac{b}{a}x} \right) \right| = \frac{1}{a} e^{-\frac{b}{a}x} \left(\frac{|r|\,e^{-x}}{1-|r|\,e^{-x}} \right), \quad \text{where}$$
$$\int_0^{\infty} \frac{1}{a} e^{-\frac{b}{a}x} \left(\frac{|r|\,e^{-x}}{1-|r|\,e^{-x}} \right) dx \leq \frac{1}{a} \int_0^{\infty} \left(\frac{|r|\,e^{-x}}{1-|r|\,e^{-x}} \right) dx < \infty.$$

This time, we apply the Lebesgue dominated convergence theorem (again, see Folland [**3**, p. 53]) to switch the order of summation and integration to obtain

$$\begin{aligned}
\sum_{n=1}^{\infty} \frac{r^n}{an+b} &= \sum_{n=1}^{\infty} \int_0^{\infty} r^n e^{-nx} (\frac{1}{a} e^{-\frac{b}{a}x})\,dx \\
&\stackrel{D.C.T.}{=} \int_0^{\infty} \left(\frac{1}{a} e^{-\frac{b}{a}x} \right) \sum_{n=1}^{\infty} r^n e^{-nx}\,dx \\
&= \frac{1}{a} \int_0^{\infty} e^{-\frac{b}{a}x} \left(\frac{re^{-x}}{1-re^{-x}} \right) dx = \frac{1}{a} \int_0^1 \frac{ru^{\frac{b}{a}}}{1-ru}\,du.
\end{aligned}$$

We have established this formula for all $r \in (-1, 1)$; but since both $\sum_{n=1}^{\infty} r^n/an+b$ and (4) exist for $r = -1$, they must also be equal by Abel's theorem (see, for instance, Buck [**1**, p. 279]).

We offer two examples. When $a = 1$ and $b = 0$, (4) yields

$$\sum_{n=1}^{\infty} \frac{r^n}{n} = \int_0^1 \frac{r}{1-ru}\,du = \ln\left(\frac{1}{1-r} \right),$$

and when $r = -1$, $a = 1$, and $b = \frac{1}{2}$, it gives

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{n+\frac{1}{2}} = -\int_0^1 \frac{u^{\frac{1}{2}}}{1+u}\,du = \frac{\pi}{2} - 2.$$

As one more example of this technique, consider

$$\sum_{n=1}^{\infty} \frac{r^n}{(n+a)(n+b)}, \quad \text{where} \quad r \in [-1, 1], \quad \text{and} \quad b > a > -1. \tag{4}$$

We already know how to write $1/((n+a)(n+b))$ as a Laplace transform integral. As long as $r \in (-1, 1)$, the estimation of partial sums is so similar to what we did before that the details are left to the reader. Once more, we apply the Lebesgue dominated convergence theorem to switch the sum and the integral, obtaining

$$\begin{aligned}
\sum_{n=1}^{\infty} \frac{r^n}{(n+a)(n+b)} &= \sum_{n=1}^{\infty} r^n \int_0^{\infty} e^{-nx} \left(\frac{e^{-ax} - e^{-bx}}{b-a} \right) dx \\
&\overset{D.C.T.}{=} \int_0^{\infty} \left(\frac{e^{-ax} - e^{-bx}}{b-a} \right) \sum_{n=1}^{\infty} r^n e^{-nx} \, dx \\
&= \frac{1}{b-a} \int_0^{\infty} (e^{-ax} - e^{-bx}) \left(\frac{re^{-x}}{1 - re^{-x}} \right) dx \\
&= \frac{r}{b-a} \int_0^1 \frac{u^a - u^b}{1 - ru} \, du.
\end{aligned} \tag{5}$$

This argument works for $r \in (-1, 1)$, but since the infinite sum and (6) both exist for $r = \pm 1$, they must be equal for $r = \pm 1$ by Abel's theorem as before. So, for instance, when $a = 0$ and $b = 1$, (6) yields

$$\sum_{n=1}^{\infty} \frac{r^n}{n(n+1)} = r \int_0^1 \frac{1-u}{1-ru} \, du = 1 + \left(\frac{1-r}{r} \right) \ln(1-r).$$

Closing remark We do not mean to suggest that the closed form expressions for the series discussed in this paper are new. There are various other ways to evaluate them. Rather, it has been our intention to give exposure to a nice technique for evaluating certain series. It is our hope that you add this technique to your toolbox of tricks for series.

Exercises Enjoy!

1. Let $b \in \{1, 2, 3, \ldots\}$. Follow the steps below to give an alternate proof of the telescoping series identity

$$\sum_{n=1}^{\infty} \frac{1}{n(n+b)} = \frac{1}{b} \left(1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{b} \right):$$

   (i) Use the Laplace transform integral $1/n + b = \int_0^{\infty} e^{-nx}(e^{-bx}) \, dx$ to show that

$$\sum_{n=1}^{\infty} \frac{1}{n(n+b)} = \int_0^1 u^{b-1} \ln \left(\frac{1}{1-u} \right) du.$$

   (ii) Show that $\int_0^1 u^{b-1} \ln(\frac{1}{1-u}) du = \frac{1}{b}(1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{b})$.

2. Find exact values for the following series, using techniques described in this paper. Check your answers with your favorite computer software package!

(i) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n(n+5)}$ (iv) $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n(n+1)}$

(ii) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n4^n}$ (v) $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n(n+2)}$

(iii) $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{2n+1}$ (vi) $\displaystyle\sum_{n=1}^{\infty} \frac{(1/2)^n}{n(n+k)}$, where $k \in \{1, 2, 3, \ldots\}$

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Answers to selected exercises from Wetzel's article *Fits and Covers*

Exercise 1 The triangle T fits into D precisely when

$$d \geq \begin{cases} \dfrac{2abc}{\sqrt{(a+b+c)(-a+b+c)(a-b+c)(a+b-c)}} & \text{if } b^2 + c^2 \geq a^2 \\ a & \text{if } b^2 + c^2 \leq a^2. \end{cases}$$

Exercise 6

(a) The disk of diameter $\sqrt{a^2 + b^2}$.

(b) The disk of diameter $2s$.

(c) The disk of diameter $p/2$.

Exercise 8 The disk of diameter 1 whose center lies at the midpoint of the curve obviously covers the curve; but so does the disk with center at the midpoint of the line segment that joins the endpoints (cf. Exercise 6(c)).

Exercise 11 Its sides are $1/(2\sqrt{3})$ and $1/\sqrt{6}$.

LINKED CITATIONS

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