These problems could make for interesting research topics for a course in game theory or for undergraduate research. We await results with anticipation.

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The Existence of Multiplicative Inverses

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It is a common in undergraduate abstract algebra courses to show that if c is square-free, the ring $\mathbb{Q}\left[\sqrt[n]{c}\right]$ is a field. The proof usually involves the notions of minimal irreducible polynomials, quotient fields, isomorphisms, and algebraic extensions (see [**1**, Ch.13] or [**2**, Ch.8], for example).

In this note, we show that every nonzero element in this ring has a multiplicative inverse, using only *basic ideas* from linear algebra and number theory.

Let c be a square-free integer. The set $\mathbb{Q}\left[\sqrt[n]{c}\right]$ is defined as

$$\left\{a_0 + a_1\sqrt[n]{c} + a_2(\sqrt[n]{c})^2 + \cdots + a_{n-1}(\sqrt[n]{c})^{n-1} | a_i \in \mathbb{Q} \text{ for all } i\right\}.$$

Let a be a nonzero element in $\mathbb{Q}\left[\sqrt[n]{c}\right]$. Our problem is to find another element $x = x_0 + x_1\sqrt[n]{c} + x_2(\sqrt[n]{c})^2 + \cdots + x_{n-1}(\sqrt[n]{c})^{n-1}$ such that $a \cdot x = 1$.

By multiplying by common denominators, we see that the problem can be restated as follows:

Let $a = a_0 + a_1\sqrt[n]{c} + a_2(\sqrt[n]{c})^2 + \cdots + a_{n-1}(\sqrt[n]{c})^{n-1}$ with $a_i \in \mathbb{Z}$. If $a \neq 0$, show that there exists an element $x = x_0 + x_1\sqrt[n]{c} + x_2(\sqrt[n]{c})^2 + \cdots + x_{n-1}(\sqrt[n]{c})^{n-1}$ with $x_i \in \mathbb{Q}$ such that $a \cdot x = m \in \mathbb{Q}$.

Furthermore, since $a \neq 0$, we can assume the coefficients of a are relatively prime, that is $\gcd(a_0, a_1, \ldots, a_{n-1}) = 1$. So, assume $a_i \in \mathbb{Z}$, and let α denote $\sqrt[n]{c}$. Multiply the two *polynomials* in the variable α, using the fact that c is square-free and $\alpha^n = c \in \mathbb{Z}$.

The existence of an inverse is then equivalent to the following condition: the coefficient of α^j is 0 for $j = 1, 2, \ldots, n-1$, and the coefficient of α^0 is some $m \in \mathbb{Q}$. We obtain a linear system of equations in the variables x_i, which written in matrix form is

$$\begin{pmatrix} a_0 & a_{n-1}c & a_{n-2}c & \cdots & a_2c & a_1c \\ a_1 & a_0 & a_{n-1}c & \cdots & a_3c & a_2c \\ a_2 & a_1 & a_0 & \cdots & a_4c & a_3c \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ a_{n-2} & a_{n-3} & a_{n-4} & \cdots & a_0 & a_{n-1}c \\ a_{n-1} & a_{n-2} & a_{n-3} & \cdots & a_1 & a_0 \end{pmatrix} \begin{pmatrix} x_0 \\ x_1 \\ x_2 \\ \vdots \\ x_{n-2} \\ x_{n-1} \end{pmatrix} = \begin{pmatrix} m \\ 0 \\ 0 \\ \vdots \\ 0 \\ 0 \end{pmatrix}.$$

Hence, a multiplicative inverse for a in $\mathbb{Q}\left[\sqrt[n]{c}\right]$ exists if and only if the determinant of the coefficient matrix A of this equation is nonzero.

Suppose $\det(A) = 0$. Since all of the entries in A are integers, this is equivalent to having $\det(A) \equiv 0 \pmod p$ for every prime p. Consider a prime p that divides c. Then

$$\det(A) \equiv \det \begin{pmatrix} a_0 & 0 & 0 & \cdots & 0 & 0 \\ a_1 & a_0 & 0 & \cdots & 0 & 0 \\ a_2 & a_1 & a_0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ a_{n-2} & a_{n-3} & a_{n-4} & \cdots & a_0 & 0 \\ a_{n-1} & a_{n-2} & a_{n-3} & \cdots & a_1 & a_0 \end{pmatrix} \equiv (a_0)^n \equiv 0 \pmod p.$$

Thus $p|a_0^n$, and hence $p|a_0$ since p is prime. This is true for every prime divisor of c, and hence $c|a_0$.

Replace a_0 by d_0c in the matrix A. We can then factor c from the first row and move the first row to the bottom of the matrix, and obtain

$$\det(A) = \det \begin{pmatrix} d_0c & a_{n-1}c & a_{n-2}c & \cdots & a_2c & a_1c \\ a_1 & d_0c & a_{n-1}c & \cdots & a_3c & a_2c \\ a_2 & a_1 & d_0 & \cdots & a_4c & a_3c \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ a_{n-2} & a_{n-3} & a_{n-4} & \cdots & d_0c & a_{n-1}c \\ a_{n-1} & a_{n-2} & a_{n-3} & \cdots & a_1 & d_0c \end{pmatrix}$$

$$= (c)(-1)^{n-1} \det \begin{pmatrix} a_1 & d_0c & a_{n-1}c & \cdots & a_3c & a_2c \\ a_2 & a_1 & d_0c & \cdots & a_4c & a_3c \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ a_{n-1} & a_{n-2} & a_{n-3} & \cdots & a_1 & d_0c \\ d_0 & a_{n-1} & a_{n-2} & \cdots & a_2 & a_1 \end{pmatrix} = 0.$$

Since $\gcd(a_0, a_1, \ldots, a_{n-1}) = 1$, it follows that $\gcd(a_1, \ldots, a_{n-1}, d_0) = 1$, and we can repeat the process with this last matrix. Therefore $c|a_1$. Continuing in the same way, we deduce that $c|a_i$, for all $i = 1, 2, \ldots, n-1$; which is a contradiction, since $\gcd(a_0, a_1, \ldots, a_{n-1}) = 1$.

Thus, the determinant of the coefficient matrix is nonzero, and a has a multiplicative inverse in $\mathbb{Q}\left[\sqrt[n]{c}\right]$.

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