Characteristic Polynomials of Magic Squares

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Square matrices which are also magic squares have many fascinating properties and provide interesting problems for the linear algebra student. Several articles which discuss properties of matrix algebra and subspaces of magic squares are listed in our references; here we make some observations about characteristic polynomials of magic squares. The main result is a consequence of a beautiful but little-known theorem of Frobenius.

A **magic square** of order n is an $n \times n$ matrix in which the entries in each row, column and diagonal sum to the same number, called the **magic sum** of the matrix. An $n \times n$ matrix whose row and column sums are the same is called a **semi-magic square**, and the common sum the **semi-magic sum**. In the theory of characteristic polynomials it is convenient to work with matrices whose entries belong to an algebraically closed field, and hence we consider the vector space $\mathbf{A}(n)$ of semi-magic squares of order n with complex entries. The set $\mathbf{M}(n)$ of magic squares of order n is a subspace of $\mathbf{A}(n)$. Since $\mathbb{C}$ is algebraically closed, the characteristic polynomial $p_A(z)$ of an $n \times n$ matrix A is a product of linear factors

$$p_A(z) = \det(zI - A) = \prod_{i=1}^{n} (z - m_i), \tag{1}$$

where the complex numbers $m_1, \ldots, m_n$ are the characteristic roots of A.

Suppose now that A is a semi-magic square of order n with semi-magic sum m. Since $Ae = me$, where $e = (1, \ldots, 1)^T$, we see that m is an eigenvalue (and hence also a characteristic root) of A. In order to easily state our results, we let $m_1 = m$ in (1), and call $m_2, \ldots, m_n$ in (1) the **complementary characteristic roots** of A. If we denote by E the magic square of order n all of whose entries are 1, then n is the magic sum of E. It is easy to see that 0 is a characteristic root of E of multiplicity $n-1$ (since rank $E = 1$, the dimension of the null space of E is $n-1$), hence the characteristic root n of E has multiplicity 1. Using our terminology, all of the complementary characteristic roots of E are 0. Note that if p is any complex number, then $A + pE$ is also semi-magic, with semi-magic sum $m + np$.

THEOREM 1. *If $A \in \mathbf{A}(n)$ and $p \in \mathbb{C}$, then A and $A + pE$ have the same complementary characteristic roots.*

The theorem is proved using the following result due to Frobenius (see [**3**], p. 22).

LEMMA (*Frobenius*). *Let A and B be $n \times n$ matrices for which $AB = BA$, let $a_1, \ldots, a_n$ be the characteristic roots of A, and let $f(x, y)$ be a rational function. The characteristic roots of B can be ordered $b_1, \ldots, b_n$ so that the characteristic roots of $f(A, B)$ are $f(a_1, b_1), \ldots, f(a_n, b_n)$.*

Proof of Theorem 1: Let $A \in \mathbf{A}(n)$ have magic sum m, and $p \in \mathbb{C}$. We can apply the Lemma with $B = E$ and $f(x, y) = x + py$, since $AE = mE = EA$. Suppose that the characteristic roots of A are $m, m_2, \ldots, m_n$. Since the semi-magic sum of $A + pE$ is $m + np$, we may order the characteristic roots of $A + pE$ as $m + np, k_2, \ldots, k_n$. Since the characteristic roots of E are $n, 0, \ldots, 0$, they can be ordered as $b_1, \ldots, b_n$ such that

$$m + np = m + pb_1$$
$$k_i = m_i + pb_i \quad \text{for } 2 \leqslant i \leqslant n.$$

From the first equation we see that $b_1 = n$. Hence the other b_i's are zero and we see that $k_i = m_i$ for $2 \leqslant i \leqslant n$ as desired.

It is well known that the determinant of a matrix is the product of its characteristic roots and thus the theorem implies

$$\frac{\det(A+pE)}{m+np}=\frac{\det A}{m}. \tag{2}$$

The reader might wish to investigate similar relationships between the sum of the principal i-rowed minors of A and of $A+pE$ (Theorem 14.3 in [**3**], p. 19 is useful).

If A is as above but is a magic square (the diagonal entries also sum to m), then we have an additional constraint. Since the trace of a matrix is the sum of its characteristic roots

$$m=\operatorname{tr}(A)=m+m_2+\cdots+m_n, \tag{3}$$

we have the following result.

THEOREM 2. *The sum of the complementary characteristic roots of* $A\in\mathbf{M}(n)$ *is zero.*

Equation (1) shows that the coefficient of z^{n-1} in the characteristic polynomial of an $n\times n$ matrix is the sum of its characteristic roots. Theorem 2 implies that this coefficient is the magic sum of A. Thus the characteristic polynomial of $A\in\mathbf{M}(n)$ with magic sum m can be written in the form

$$(z-m)\left(z^{n-1}+a_{n-3}z^{n-3}+\cdots+a_0\right). \tag{4}$$

Our results can be applied to low-order matrices to provide interesting exercises. We illustrate for $n=3$. In [**6**] it is proved that $\dim\mathbf{M}(3)=3$, and it is a nice exercise to show that the matrices E, F, G, where

$$F=\begin{bmatrix}1&-1&0\\-1&0&1\\0&1&-1\end{bmatrix}\text{ and } G=\begin{bmatrix}0&1&-1\\-1&0&1\\1&-1&0\end{bmatrix},$$

is a basis for $\mathbf{M}(3)$. If $A\in\mathbf{M}(3)$, then A can be written uniquely in the form

$$A=aE+bF+cG,\qquad a,b,c\in\mathbb{C}$$

and one can easily see that the magic sum of A is $m=3a$ and

$$\det A=9a(c^2-b^2). \tag{5}$$

If A' denotes the matrix of cofactors of A, a direct calculation shows that

$$A'=(c^2-b^2)E-3abF+3acG,$$

so $A'\in\mathbf{M}(3)$. But then $m'=3(c^2-b^2)$ is the magic sum of A' and (5) shows that $mm'=\det A$. If A is invertible, then $A^{-1}=(1/\det A)A'$, so $A^{-1}\in\mathbf{M}(3)$ and the magic sum of A^{-1} is $1/m$. This gives an alternate proof of results of Rose [**4**] and Lancaster [**1**], which state respectively that if A is an invertible magic square of order three, then A^{-1} is also a magic square of order three and the magic sum of A^{-1} is the reciprocal of the magic sum of A.

As a consequence of (2) and (4) we have: if $A\in\mathbf{M}(3)$, then, in the preceding notation, the characteristic polynomial of A is $(z-m)(z^2-m')$.

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