45th United States of America Mathematical Olympiad

Day I  12:30PM — 5PM EDT
April 19, 2016

Note: For any geometry problem, the first page of the solution must be a large, in-scale, clearly labeled diagram made with drawing instruments (ruler, compass, protractor, graph paper). Failure to meet this requirement will result in an automatic 1-point deduction.

**USAMO 1.** Let $X_1, X_2, \ldots, X_{100}$ be a sequence of mutually distinct non-empty subsets of a set $S$. Any two sets $X_i$ and $X_{i+1}$ are disjoint and their union is not the whole set $S$, that is, $X_i \cap X_{i+1} = \emptyset$ and $X_i \cup X_{i+1} \neq S$, for all $i \in \{1, \ldots, 99\}$. Find the smallest possible number of elements in $S$.

**USAMO 2.** Prove that for any positive integer $k$,

$$
(k^2)! \cdot \prod_{j=0}^{k-1} \frac{j!}{(j+k)!}
$$

is an integer.

**USAMO 3.** Let $\triangle ABC$ be an acute triangle, and let $I_B$, $I_C$, and $O$ denote its $B$-excenter, $C$-excenter, and circumcenter, respectively. Points $E$ and $Y$ are selected on $\overline{AC}$ such that $\angle ABY = \angle CBY$ and $\overline{BE} \perp \overline{AC}$. Similarly, points $F$ and $Z$ are selected on $\overline{AB}$ such that $\angle ACZ = \angle BCZ$ and $\overline{CF} \perp \overline{AB}$.

Lines $\overrightarrow{IF}$ and $\overrightarrow{IC}$ meet at $P$. Prove that $\overline{PO}$ and $\overline{YZ}$ are perpendicular.