Carl B. Allendoerfer Awards

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“What (quilting) circles can be squared?” *Mathematics Magazine*, 92(3), 173–186.
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For evidence that mathematics—and the origin of mathematical journeys—can arise in unexpected places, one needs look no farther than this article. The story begins with a question from a friend about a quilting circle: how can one arrange five rounds of quilt hand-offs among a group of five quilters so that every quilter hands off once to every other quilter? That way each quilter gets to meet every other one, rather than, say, always passing to the same person. The friend is not quite able to get this to work, and so appeals to Prof. Malmskog. As the authors write, “who could resist”?

It turns out the quilting hand-offs can be represented in the form of a Latin square. We assign each quilter a number, using the numbers 0, 1,…, $n - 1$ for n quilters. The quilters are arranged into the left column of the Latin square. The quilters they hand-off-to appear in subsequent columns. A sample Latin square for four quilters is shown in Figure 1. Notice that the first row indicates quilter 0 hands off to quilter 1, who hands off to quilter 3, who hands off to quilter 2.

0	1	3	2
1	2	0	3
2	3	1	0
3	0	2	1

Figure 1: A Latin square for four quilters

The authors’ goal is to find not just any Latin square, but one in which all the hand-offs are different, so that each quilter hands off to each other quilter. The hand-offs from one quilter to another correspond to the sequences (i, j) in each row of the Latin square. For instance, in the first row of Figure 1, the sequences are $(0,1)$, $(1,3)$, and $(3,2)$. To ensure that every quilter hands off once to every other quilter, we require that each pair (i, j) appear one and only one time when reading across rows. Latin squares with this additional property are called row-complete, and the Latin square in Figure 1 is an example. In addition to their desirability for quilting circles, row-complete Latin squares play a role in the design of experiments in which treatments might have residual effects, such as taste-testing experiments.

Surprisingly, no row-complete Latin squares of order 5 exist, as the authors show using a case analysis. So there was a good reason the first author’s friend couldn’t get things to work! The question of determining whether there is a row-complete Latin square of given size n is a deep one, which remains unresolved in full generality. Constructions for even n date from the 1940s, and constructions for odd composite n date from the 1990s. The case of prime n, aside from $n = 3,5,7$, remains mysterious.

You may notice that in the example (1), the sequence of successive differences of elements in each row, considered modulo 4, is always 1, 2, 3. Such row-complete Latin squares are called rotational, and it’s possible to find them whenever n is even. Interestingly, the existence of such Latin squares is equivalent to the cyclic group of order n having a group-theoretic property called sequenceability, and a theorem from the 1960s shows that this holds only for even n.

The article closes by introducing "quiltdoku" problems: can a partially filled grid be completed to become a row-complete Latin square? A new fabric-themed puzzling craze could be upon us. The authors let us accompany them on their journey with clarity, gentleness, and efficiency. By the end, we've visited graph theory, group theory, combinatorics, and even experimental design—a pleasing patchwork of mathematical ideas.

Response

We are thrilled and honored to receive the Carl B. Allendoerfer Award! From the question that sparked our inquiry through our continuing investigations today, the work chronicled in this paper has been about building connection and community through mathematics. Even the original quilting question came to us through friendship and the surprising reach of a mathematics puzzle segment on a community radio show in Colorado—this great question was one of the perks of being known as the radio math lady within a tiny world. This award is especially wonderful to us in this context, because it lets us know that we have connected with the larger world in a meaningful way. Thank you to the awards committee and to the MAA!

Biographical Sketch

Beth Malmskog wishes more of her friends would turn their hobbies into math problems for her to work on. Dr. Malmskog received her PhD from Colorado State University in 2011. Her research is in number theory and discrete mathematics, including computational questions and applications. She is now an assistant professor in the Department of Mathematics and Computer Science at Colorado College.

Katie Haymaker obtained a PhD in mathematics from the University of Nebraska-Lincoln in 2014. She is an associate professor in the Department of Mathematics and Statistics at Villanova University. Dr. Haymaker's research focuses on coding theory and applied discrete mathematics. She is grateful that Dr. Malmskog included her in discussions of this fascinating problem, which has led to many exciting discoveries and re-discoveries.