The Harmonic Triangle and the Beta Function

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The Harmonic Triangle

$$\begin{array}{cccccc} 1 & \frac{1}{2} & \frac{1}{3} & \frac{1}{4} & \frac{1}{5} & \cdot \\ & \frac{1}{2} & \frac{1}{6} & \frac{1}{12} & \frac{1}{20} & \cdot \\ & & \frac{1}{3} & \frac{1}{12} & \frac{1}{30} & \cdot \\ & & & \frac{1}{4} & \frac{1}{20} & \cdot \\ & & & & \frac{1}{5} & \cdot \\ & & & & & \cdot \end{array}$$

is the difference table of the harmonic sequence. Although this triangle is not as widely known as Pascal's triangle, it has been around for quite a while. In fact, its origin can be traced back to the seventeenth century when Leibniz used the sequential differences in summing some infinite series [**1**], [**2**], and [**4**]. For this reason the harmonic triangle is sometimes called Leibniz' triangle. Here, we will show that the entries of this array are the values of the beta function

$$\beta(\mu,\nu)=\int_0^1 x^{\mu-1}(1-x)^{\nu-1}\,dx$$

for the integral parameters μ, ν with $\mu \geq 1$, $\nu \geq 1$.

Using this fact, we will be able to justify the symmetry within the harmonic triangle and to find row sums and column sums in this triangle. Also, we will derive the basic relationship between the harmonic triangle and Pascal's triangle.

If we number the columns and rows starting with zero, then the entry in the nth column and rth row—which we denote by H_r^n—is the difference $\Delta^r u_{n-r}$. A "well-known" difference formula [**3**] tell us that:

$$\Delta^r u_{n-r}=u_{n-r}-\binom{r}{1}u_{n-r+1}+\binom{r}{2}u_{n-r+2}-\cdots+(-1)^r u_n,$$

where the u_n's are terms in the zero row. Hence,

$$H_r^n=\frac{1}{n-r+1}-\binom{r}{1}\frac{1}{n-r+2}+\binom{r}{2}\frac{1}{n-r+3}-\cdots+(-1)^r\frac{1}{n+1}.$$

But

$$\int_0^1 x^{n-r}(1-x)^r\,dx=\int_0^1\left[x^{n-r}-\binom{r}{1}x^{n-r+1}+\binom{r}{2}x^{n-r+2}-\cdots+(-1)^r x^n\right]dx$$

$$=\frac{1}{n-r+1}-\binom{r}{1}\frac{1}{n-r+2}+\binom{r}{2}\frac{1}{n-r+3}-\cdots+(-1)^r\frac{1}{n+1}.$$

Thus, $H_r^n=\int_0^1 x^{n-r}(1-x)^r\,dx$ for all integers $n \geqslant r \geqslant 0$. And by the definition of the beta function, we have

$$H_r^n=\beta(n-r+1,r+1)\quad\text{for all integers } n\geqslant r\geqslant 0.$$

Now we may conclude the following:

(1) The symmetry within each column, namely $H_r^n = H_{n-r}^n$ is due to the equality

$$\beta(n-r+1, r+1) = \beta(r+1, n-r+1),$$

each side being equal to $\Gamma(n-r+1)\Gamma(r+1)/\Gamma(n+2)$.

(2) If $r \geqslant 1$, then the sum of the entries in the rth row is

$$\sum_{k=r}^{\infty} H_r^k = \sum_{k=r}^{\infty} \int_0^1 x^{k-r}(1-x)^r \, dr = \int_0^1 \frac{1}{1-x}(1-x)^r \, dx$$
$$= \int_0^1 (1-x)^{r-1} \, dx.$$

Thus,

$$\sum_{k=r}^{\infty} H_r^k = H_{r-1}^{r-1} \quad \text{for all } r \geqslant 1.$$

As an example, the sum of the entries in the second row is

$$\frac{1}{3} + \frac{1}{12} + \frac{1}{30} + \cdots = \frac{1}{2}.$$

(3) To find the sum of the entries in the nth column, we have

$$\sum_{r=0}^{n} H_r^n = \sum_{r=0}^{n} \int_0^1 x^{n-r}(1-x)^r \, dx = \int_0^1 \frac{x^{n+1} - (1-x)^{n+1}}{x - (1-x)} \, dx.$$

Using the transformation $x = \frac{1}{2} + y$, we get

$$\sum_{r=0}^{n} H_r^n = \left(\frac{1}{2}\right)^n \sum_{r=0}^{\left[\frac{n}{2}\right]} \binom{n+1}{2r+1} \Big/ (2r+1).$$

(4) Since

$$H_r^n = \beta(n-r+1, r+1) = \frac{\Gamma(n-r+1)\Gamma(r+1)}{\Gamma(n+2)} = \frac{(n-r)!r!}{(n+1)!},$$

then

$$H_r^n = \frac{1}{(n+1)\binom{n}{r}}.$$

This equality gives us the relationship between the harmonic triangle and Pascal's triangle

1	1	1	1	1	•
	1	2	3	4	•
		1	3	6	•
			1	4	•
				1	•

(5) If we use the previous relation together with the column sum in (3) we get the identity

$$\frac{1}{n+1} \sum_{r=0}^{n} \binom{n}{r}^{-1} = \left(\frac{1}{2}\right)^n \sum_{r=0}^{\left[\frac{n}{2}\right]} \binom{n+1}{2r+1} \Big/ (2r+1),$$

or, equivalently,

$$\sum_{r=0}^{n} \binom{n}{r} \cdot \sum_{r=0}^{n} \binom{n}{r}^{-1} = (n+1) \sum_{r=0}^{\left[\frac{n}{2}\right]} \binom{n+1}{2r+1} \Big/ (2r+1).$$

Also, the relation in (4) implies that $\sum_{r=0}^{n} \binom{n}{r} H_r^n = 1$.

References

[1] M. Bicknell-Johnson, Diagonal sums in the harmonic triangle, The Fibonacci Quarterly, 19 (1981) 196–198.
[2] C. B. Boyer, A History of Mathematics, John Wiley & Sons, Inc, N.Y., 1968, p. 439–440.
[3] G. Chrystal, Algebra, Part II, Chelsea Publishing, N.Y., 1964, p. 401.
[4] G. Pólya, Mathematical Discovery, John Wiley & Sons, Inc., N.Y., 1981, p. 89.

A Note on Evaluating Limits Using Riemann Sums

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In this note, we present a method for evaluating some limits. This method, which does not appear to be widely used, involves expressing the limit in the form of a Riemann sum. This sum, in turn, is equivalent to a definite integral, which can be evaluated using standard techniques. We illustrate the technique with several examples.

The first example appears as an exercise in the book *Advanced Calculus* by David V. Widder [2, p. 391]. It can also be solved using Stirling's formula or series methods.

Example 1. Evaluate

$$\lim_{n\to\infty}\left(\frac{1}{n}\ln n! - \ln n\right).$$

Solution. We have

$$\begin{aligned}\lim_{n\to\infty}&\left(\frac{1}{n}\ln n! - \ln n\right)\\ &= \lim_{n\to\infty}\left[\frac{1}{n}\left(\sum_{k=1}^{n}\ln k\right) - \ln n\right]\\ &= \lim_{n\to\infty}\frac{1}{n}\left[\sum_{k=1}^{n}(\ln k - \ln n)\right]\\ &= \lim_{n\to\infty}\frac{1}{n}\sum_{k=1}^{n}\ln\left(\frac{k}{n}\right).\end{aligned}$$

This limit is the Riemann sum of $f(x) = \ln x$ over the interval $[0,1]$. Hence,

$$\lim_{n\to\infty}\left(\frac{1}{n}\ln n! - \ln n\right) = \int_0^1 \ln x\,dx.$$

Integrating this improper integral by parts, and using l'Hôpital's rule, we get

$$\int_0^1 \ln x\,dx = -1.$$